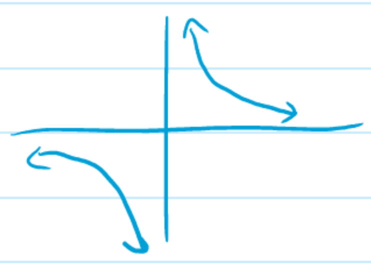
Tutorial 1 Solutions

January 11, 2025 11:29 AM

Agenda: * TA introduction
* Course logistics
↳ use teams!

Similar to: HW1 Problem 5

$\int_0^6 \frac{e^{1/x}}{x^2}\,dx$ try: u-substitution



$u = x^{-1} = \frac{1}{x}$ $du = -x^{-2}dx = \frac{-1}{x^2}dx$ (see graph)

as $x \to 6$ $u \to \frac{1}{6}$ as $x \to 0^+$ $u \to \infty$ (from the right.)

$$\int_0^6 \frac{e^{1/x}}{x^2}\,dx = \int_\infty^{1/6} e^u(-du) = \int_{1/6}^\infty e^u\,du = e^u + C \Big|_{1/6}^\infty$$

$$= \lim_{u\to\infty} e^u + c - (e^{1/6} + c)$$

∞

hence this is divergent.

Similar to: HW1 Problem 6

do these converge?

(i) $\int_1^\infty \frac{\arctan x}{e^x+2}\,dx$

Note: $\arctan(x) \le \pi/2$
since $e^x + 2 \ge e^x \Rightarrow \frac{1}{e^x+2} \le \frac{1}{e^x}$
} $\frac{\arctan(x)}{e^x+2} \le \frac{\pi/2}{e^x}$

$$\int_1^\infty \frac{\pi}{2}e^{-x}\,dx = \frac{\pi}{2}\left(-e^{-x}+c\Big|_1^\infty\right) = \frac{\pi}{2}\left(\lim_{x\to\infty} -e^{-x}+c - (-e^{-1}+c)\right)$$

$$= \frac{\pi}{2}\left(0 + c - (-e^{-1}+c)\right) = \frac{\pi}{2} + e^{-1} \text{ is finite.}$$

Hence $\int_1^\infty \frac{\arctan(x)}{e^x+2}\,dx$ converges.

(ii) $\int_2^\infty \frac{\sqrt{x^8-2}}{x^6}\,dx$ $\frac{x^8-2}{x^6} < \frac{x^8}{x^6} \Rightarrow \frac{\sqrt{x^8-2}}{x^6} < \frac{x^4}{x^6} = x^{-2}$

$$\int_2^\infty x^{-2}\,dx = \frac{x^{-1}}{-1} + c\Big|_2^\infty = \lim_{x\to\infty} \frac{-1}{x} + c - \left(\frac{-1}{2} + c\right) = \frac{1}{2}$$

which converges. Hence, $\int_2^\infty \frac{\sqrt{x^8-2}}{x^6}\,dx$ converges.

(iii) $\int_0^{\pi/2} \frac{2-\sin(x)}{x^{1/2}}\,dx$ $0 \le \sin(x) \le 1$ hence $\frac{2-\sin(x)}{x^{1/2}} \le \frac{3}{x^{1/2}}$

$$\int_0^{\pi/2} 3x^{-1/2}\,dx = 3(2x^{1/2} + c)\Big|_0^{\pi/2} = 3(2(\pi/2)^{1/2}) = 3\sqrt{2\pi}$$

hence this converges.

Similar to HW2 Problem 2:

find the limit of $\{a_n\}$ where $a_1 = 1$, $a_{n+1} = \frac{8a_n}{1+a_n}$,
assuming $\{a_n\}$ converges.

Note: "a_n converges" means there exists ("$\exists\ L \in \mathbb{R}$") a finite number L such that $\lim_{n\to\infty} a_n = L$.

intuitively, if $\lim_{n\to\infty} a_n = L$ then $\lim_{n\to\infty} a_{n+1} = L$ as well.

So, $a_{n+1} = \frac{8a_n}{1+a_n} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{8a_n}{1+a_n} = \frac{8 \lim_{n\to\infty} a_n}{1 + \lim_{n\to\infty} a_n}$ (each limit $= L$)

$$\Rightarrow L = \frac{8L}{1+L} \Rightarrow L(1+L) = 8L \Rightarrow L + L^2 - 8L = 0 \Rightarrow L^2 - 7L = 0$$

$L(L-7) = 0$
↓
$L = 0$ or $L = 7$

Observe the behaviour: $a_1 = 1$ (incr) $a_2 = \frac{8(1)}{1+1} = 4$ (incr) $a_3 = \frac{8(4)}{1+(4)} = \frac{32}{5} = 6.4$

WTShow: a_n is incr using induction, i.e, $a_{n+1} > a_n$.

Base Case: $n = 2$, $a_2 = 4 > 1 = a_1$ ✓

Induction Hypothesis: $\exists k \in \mathbb{N}$ s.t $a_k > a_{k-1}$
"there exists a natural number k such that the sequence increases."

Induction Step: WTShow: $a_{k+1} \ge a_k$.

Note that $a_{k+1} = \frac{8a_k}{1+a_k}$ $a_k = \frac{8a_{k-1}}{1+a_{k-1}}$

we can work backwards:

WTShow: $\frac{8a_k}{1+a_k} \ge \frac{8a_{k-1}}{1+a_{k-1}}$

$\Longleftrightarrow$ $a_k(1+a_{k-1}) \ge a_{k-1}(1+a_k)$

$\Longleftrightarrow$ $a_k + \cancel{a_k a_{k-1}} \ge a_{k-1} + \cancel{a_{k-1}a_k}$

$\Longleftrightarrow$ $a_k \ge a_{k-1}$ which is true from the IH.

SINCE a_n is incr and $a_1 = 0$ then $\lim_{n\to\infty} a_n \ne 0$, as $0 < 1$.

Thus $\lim_{n\to\infty} a_n = 7$.

HW2 Q5 — do this