Sample Test 3 Version 1 Answers

March 31, 2025 8:53 PM

Plan: Q3,10,12

Convert the polar equation $r = 2(1-\cos\theta)^{-1}$ to Cartesian coordinates
(a) $x = y^2+2$ **(b)** $x^2+y^2=2$ **(c)** $y = 4x+3$ **(d)** $x = \frac{1}{4}(y^2-4)$ **(e)** $x = 4y+4$

$r = \frac{2}{1-\cos\theta}$ Recall: $r\cos\theta = x$

$r = \frac{2r}{r - r\cos\theta} = \frac{2r}{r-x}$

$\Rightarrow r^2 - rx = 2r$

$\Rightarrow r - x = 2$

$\Rightarrow \sqrt{y^2+x^2} - x = 2$

$\Rightarrow y^2 + x^2 = (2+x)^2$

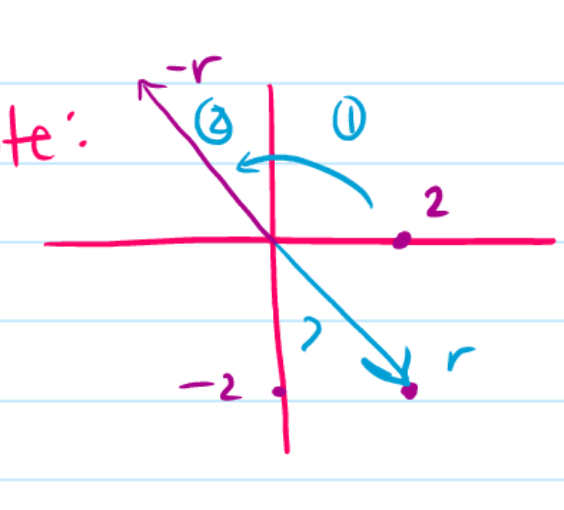
$\Rightarrow y^2 + \cancel{x^2} = 4 + 4x + \cancel{x^2}$

$\Rightarrow \frac{y^2-4}{4} = x$

Final answer: $x = \frac{1}{4}(y^2-4)$ (d)

2. Find a set of polar coordinates for $(2,-2)$ with negative r.
(a) $(-2\sqrt{2}, \frac{\pi}{4})$ **(b)** $(-2\sqrt{2}, \frac{3\pi}{4})$ **(c)** $(-2\sqrt{2}, -\frac{\pi}{4})$ **(d)** $(-2\sqrt{2}, -\frac{3\pi}{4})$
(e) $(-2\sqrt{2}, -\frac{3\pi}{2})$

note:



$r^2 = \sqrt{x^2+y^2}$

$= \sqrt{4+4} = \sqrt{2(4)}$

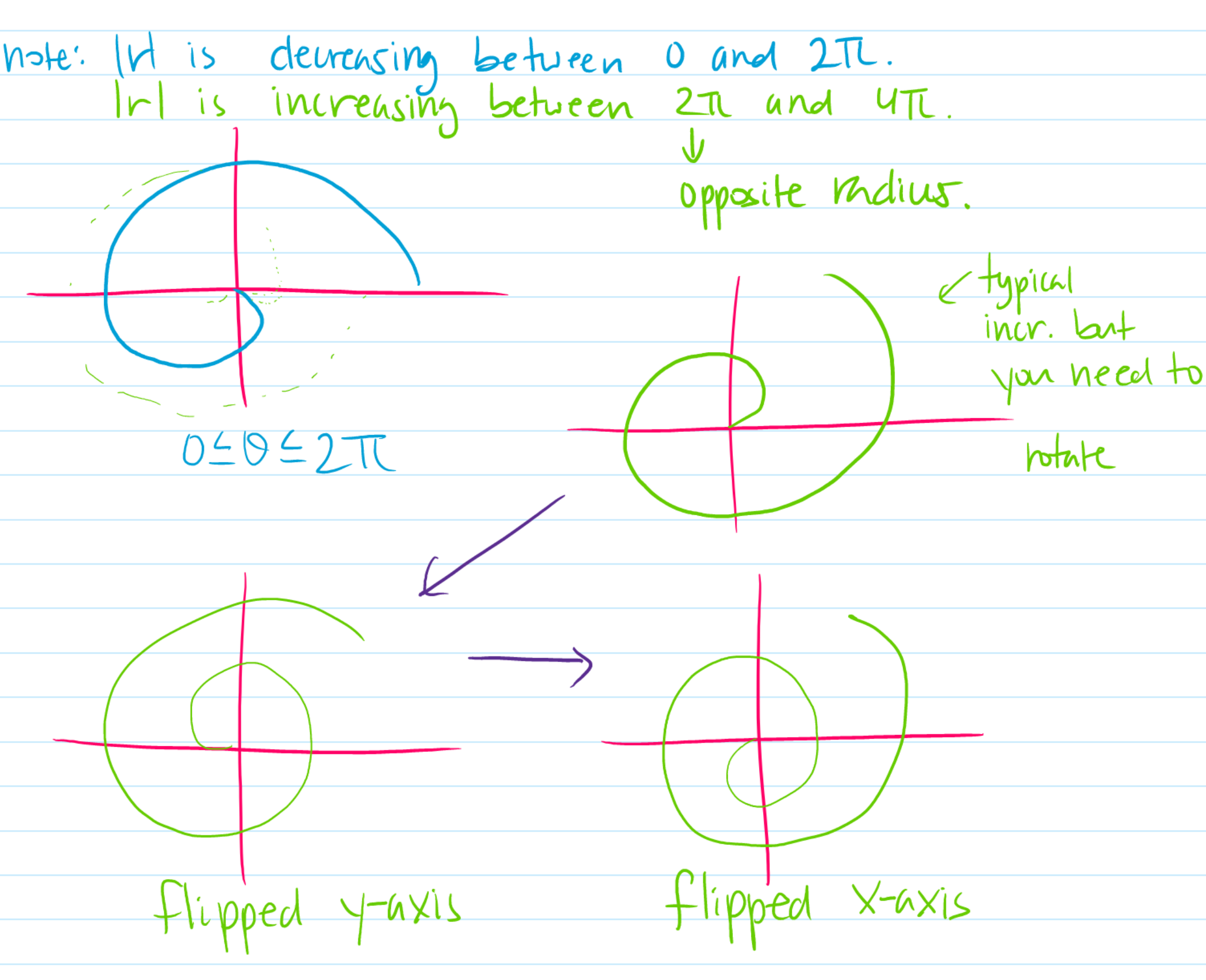
$= 2\sqrt{2}$

①: $\pi/2$ ②: $\pi/4$ thus $\pi/2 + \pi/4 = 3\pi/4$ (b)

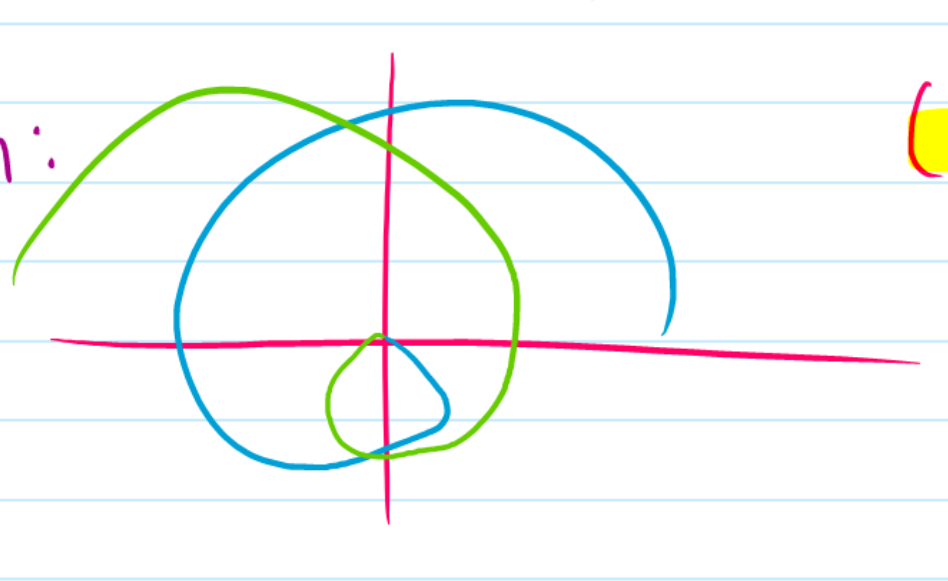
3. Sketch the polar curve $r = -\theta + 2\pi$, $0 \le \theta \le 4\pi$.

$r = -\theta + 2\pi$ if $\theta = 0$ then $r = 2\pi$ $\theta = 2\pi$: $r = 0$
$\theta = 4\pi$ then $r = -2\pi$

note: $|r|$ is decreasing between 0 and 2π.
$|r|$ is increasing between 2π and 4π.
$\downarrow$
opposite radius.



Combine them:



(d)

4. Evaluate the following limit,

$$\lim_{(x,y)\to(0,0)} \frac{xye^y}{x^4+4y^2}$$

(a) 0 **(b)** d.n.e. **(c)** 1 **(d)** $\frac{1}{2}$ **(e)** $\frac{1}{4}$

1. always check if the limit exists...

Sanity checks:

when $x=0$: $f(0,y) = \frac{0}{4y^2} = 0$ $y=0$: $f(x,y) = 0$

$x=y$: $f(x,x) = \frac{x^2e^x}{x^4+4x^2} = \frac{e^x}{x^2+4} \xrightarrow{x\to 0} \frac{1}{4}$ not consistent w/ $x=0$ or $y=0$.

DNE (b)

5. Let $w(s,t) = F(u(s,t), v(s,t))$, where F, u, and v are differentiable, $u(1,0) = 2$, $u_s(1,0) = -2$, $u_t(1,0) = 6$, $v(1,0) = 3$, $v_s(1,0) = 5$, $v_t(1,0) = 4$, $F_u(2,3) = -1$, and $F_v(2,3) = 10$. Find $w_t(1,0)$.
(a) 41 **(b)** 5 **(c)** 16 **(d)** 28 **(e)** 34

Sanity check:
$u(1,0) = 2$
$v(1,0) = 3$

$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial t}$

$= F_u(2,3)\cdot u_t(1,0) + F_v(2,3)\cdot v_t(1,0)$

$= (-1)\cdot 6 + (10)\cdot 4$

$= \boxed{34}$ (e)

6. Which of the following differential equations is satisfied by the function $u = \ln\sqrt{x^2+y^2}$?
(a) $u_{xy} - u_{xx} = 0$ **(b)** $2u_{xy} - u_{xx} = 0$ **(c)** $u_{xx} + u_{yy} = 0$
(d) $u_{xx} - u_{yy} = 0$ **(e)** $2u_{xx} - u_{yy} = 0$

$u = \ln((x^2+y^2)^{1/2})$ Check and guess...

$u = \frac{1}{2}\ln(x^2+y^2)$

$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$

$u_{xx} = \frac{(1)(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

note: u_{yy} is similar $u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$ (notice the pattern?)

Thus $u_{xx} + u_{yy} = \frac{(y^2-x^2)+(x^2-y^2)}{(x^2+y^2)^2} = 0$

Thus (c)

7. Find the linearization of $f(x,y) = y\ln x$ at $(2,1)$.
(a) $\frac{1}{2}x + (\ln 2)y - 1$ **(b)** $\frac{1}{2}x - (\ln 2)y - 1$ **(c)** $\frac{1}{2}x + (\ln 2)y + 1$
(d) $2x - 1$ **(e)** $2x + (\ln 2)y - 1$

$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f_x(x,y) = \frac{y}{x}$ $f_y(x,y) = \ln x$

$f_x(2,1) = \frac{1}{2}$ $f_y(2,1) = \ln(2)$

$= \ln(2) + \frac{1}{2}(x-2) + \ln(2)(y-1)$

$= \cancel{\ln(2)} + \frac{x}{2} - 1 + \ln(2)y - \cancel{\ln(2)}$

$= \frac{x}{2} + \ln(2)y - 1$ (a)

8. Find the equation of the tangent plane to the surface $z = 5 - x^2 + y^2$ at the point $(-2,1,2)$.
(a) $z = 4x + 2y + 8$ **(b)** $z = 4x + 2y - 8$ **(c)** $z = 4x - 2y + 8$
(d) $z = 4x - 2y - 8$ **(e)** $z = 4x - 2y + 6$

Consider $x^2 - y^2 + z = 5$

$F(x,y,z) = x^2 - y^2 + z$

$F_x = 2x$ $F_y = -2y$ $F_z = 1$

$F_x(-2,1,2) = -4$ $F_y(-2,1,2) = -2$

Thus: $F_x(-2,1,2)(x+2) + F_y(-2,1,2)(y-1) + F_z(-2,1,2)(z-2) = 0$

$-4x - 8 - 2y + \cancel{2} + z - \cancel{2} = 0$

$-4x - 2y - 8 + z = 0$

$z = 4x + 2y + 8$ (a)

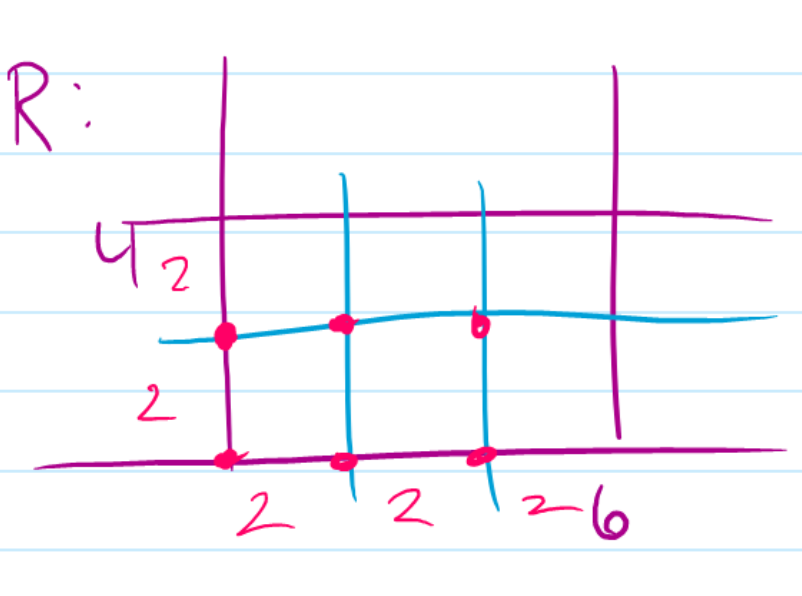
9. Estimate the volume of the solid that lies below the surface $z = xy$ and above the rectangle

$$R = \{(x,y)\,|\,0 \le x \le 6,\ 0 \le y \le 4\}$$

Use a Riemann sum with $m = 3$, $n = 2$, and take the sample point to be the lower left corner of ~~each square~~.
(a) 32 **(b)** 24 **(c)** 16 **(d)** 48 **(e)** 96

double Riemann sum : $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$

R:



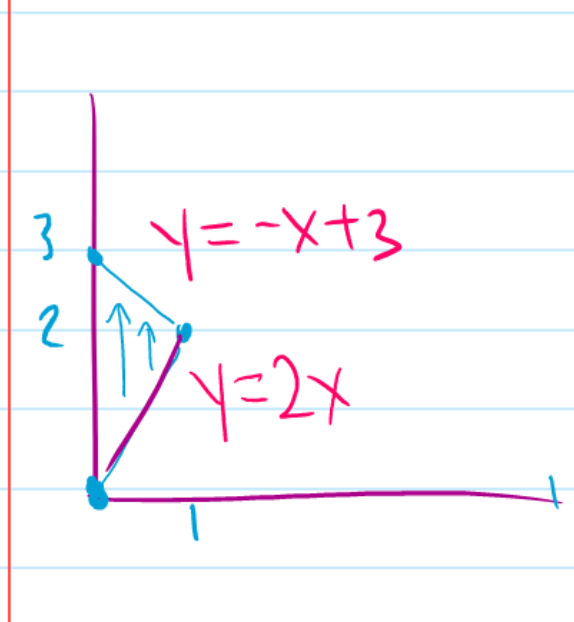
divide x regions by m
divide y regions by
$\Delta A = \Delta x\Delta y = (2)(2) = 4$

Consider points $(0,0)$, $(2,0)$, $(4,0)$
$(0,2)$, $(2,2)$, $(4,2)$

$f(0,0) = 0$ $f(0,2) = 0$
$f(2,0) = 0$ $f(2,2) = 4$
$f(4,0) = 0$ $f(4,2) = 8$

Thus, final sum: $(4+8)(4)$
$= \boxed{48}$ (d)

10. Set up a double integral which represents the volume under the function $z = 2xy$ and above the triangular region with vertices $(0,0)$, $(1,2)$, and $(0,3)$.
(a) $\int_0^1\int_x^{-x+3} 2xy\,dy\,dx$ **(b)** $\int_0^1\int_{2x}^{3} 2xy\,dy\,dx$ **(c)** $\int_0^1\int_0^{-x+3} 2xy\,dy\,dx$
(d) $\int_0^1\int_{2x}^{-x} 2xy\,dy\,dx$ **(e)** $\int_0^1\int_{2x}^{-x+3} 2xy\,dy\,dx$



options imply you must do it w.r.t y

$\int_0^1\int_{2x}^{-x+3} 2xy\,dy\,dx$ (e)

$y = mx$

$(2) = m(1) \rightarrow m = 2$

$m = \frac{\text{rise}}{\text{run}} = \frac{1}{-1} = -1$

$(3) = (-1)(0) + b$

$y = -x + 3$

11. Evaluate

$$\int_0^2\int_0^1 ye^{xy}\,dydx$$

(a) $\frac{1}{2}(e^2-3)$ **(b)** $\frac{1}{4}(e^2+2)$ **(c)** e^2-1 **(d)** e^2+3 **(e)** $\frac{1}{3}(e^2-1)$

$\int_0^2\int_0^1 ye^{xy}\,dydx = \int_0^1\int_0^2 ye^{xy}\,dxdy$

$= \int_0^1 y\underline{\int_0^2 e^{xy}\,dx}\,dy$

$\frac{e^{xy}}{y}\Big|_0^2 = \frac{e^{2y}}{y} - \frac{1}{y}$

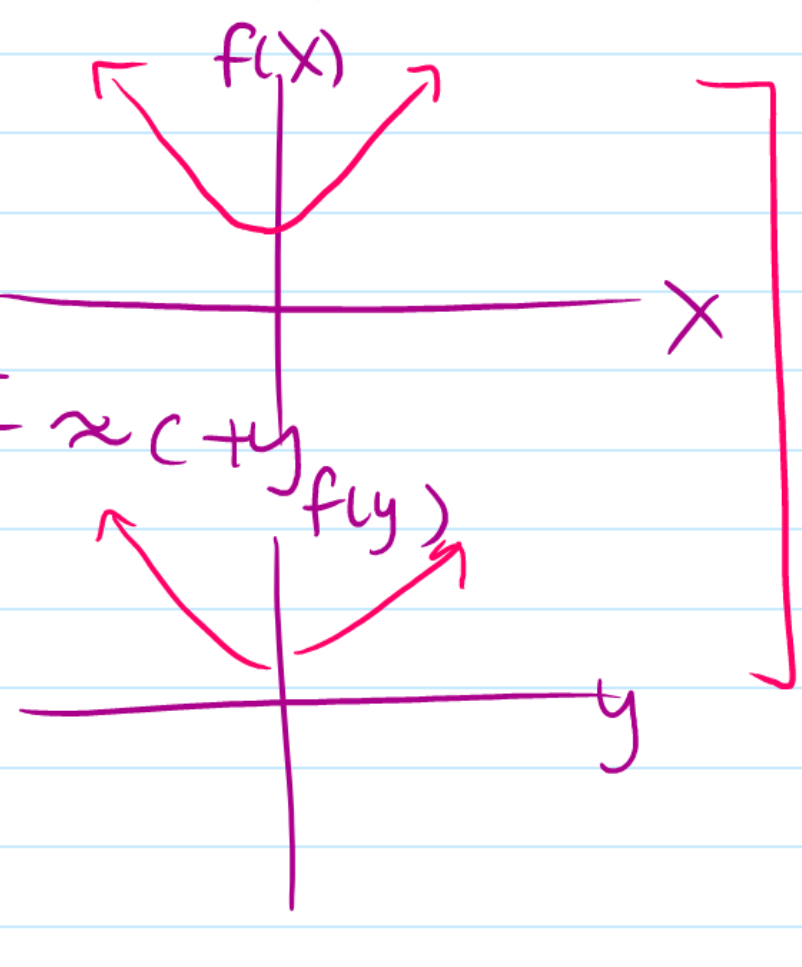
$= \int_0^1 y\left(\frac{e^{2y}-1}{y}\right)dy = \int_0^1 e^{2y} - 1\,dy$

$= \frac{e^{2y}}{2} - y\Big|_0^1 = \frac{e^2}{2} - 1 - \left(\frac{1}{2}\right) = \frac{e^2}{2} - \frac{3}{2}$ (a)

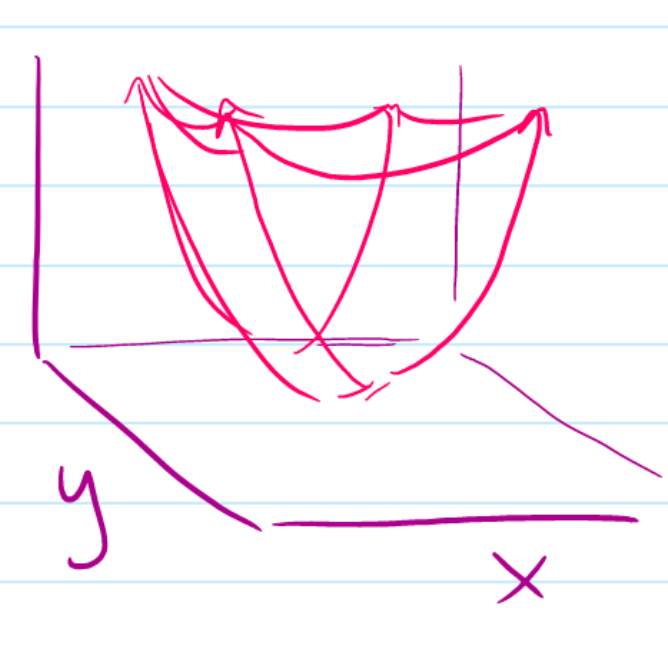
12. Sketch the graph of the function $f(x,y) = \sqrt{x^2+y^2}$

first: keep y constant. $f(x) = \sqrt{x^2+c^2} \approx x + c$ (Approx!)
ie, $y = c$ looks like:

Second: Keep x constant: $f(y) = \sqrt{c^2+y^2} \approx c + y$
have: shape that is always positive.



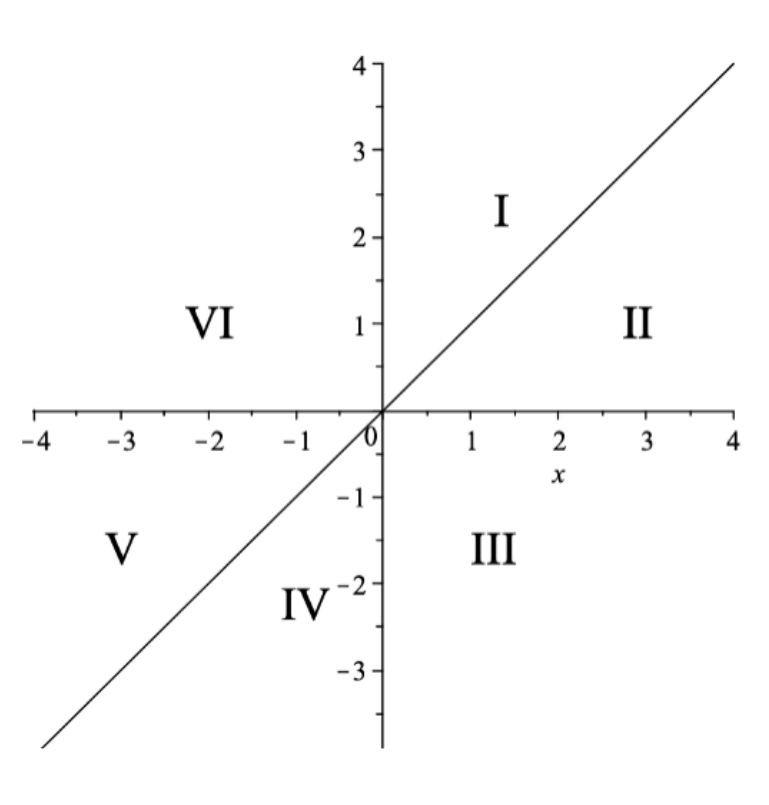
implies "V" shape



Closest to (d)

13. Which of the regions in the figure to the right represents the domain of $f(x,y) = \ln(xy) + \sqrt{x-y}$?

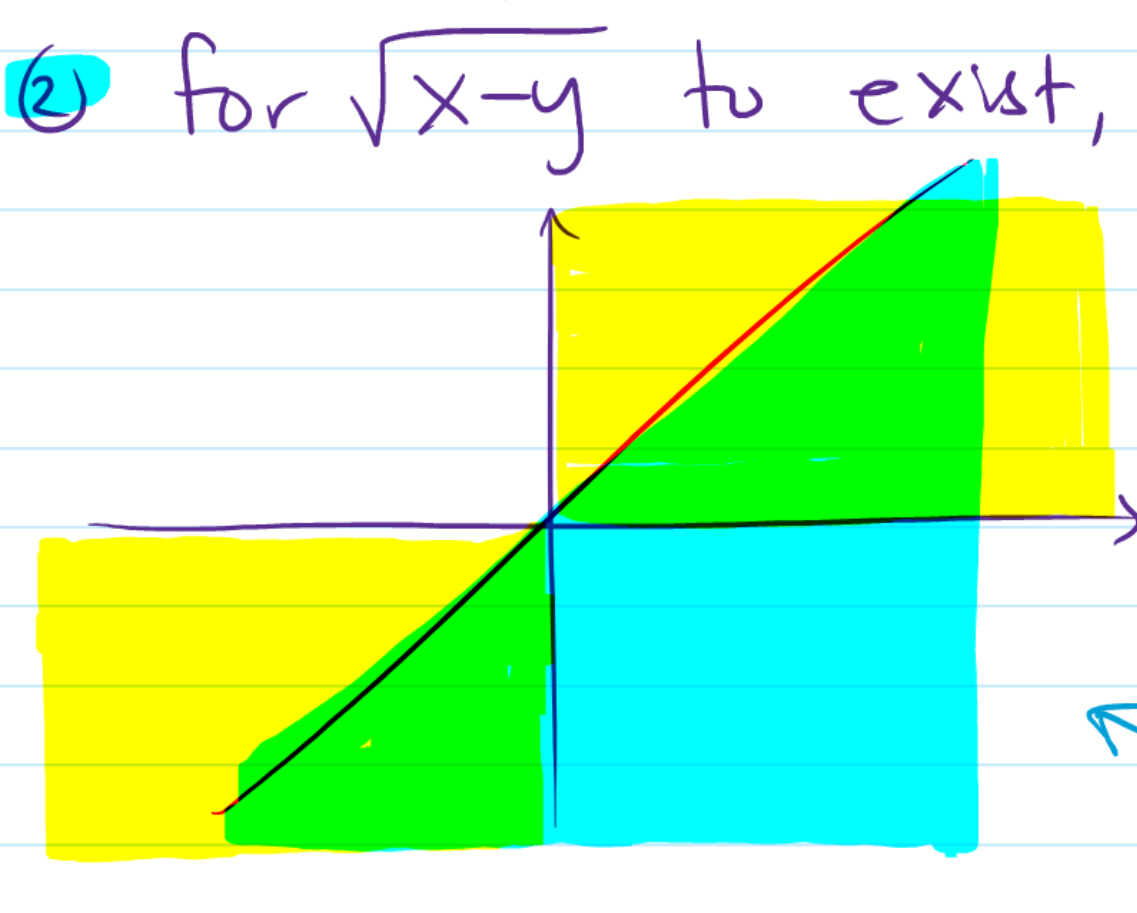
(a) Regions II, III, and IV
(b) Regions II and IV
(c) Reions II and V
(d) Regions I and IV
(e) Regions I, VI, and V



Domain:

① for $\ln(xy)$ to exist, $xy > 0$ so either
$x > 0$ & $y > 0$ or $x < 0$ & $y < 0$.

② for $\sqrt{x-y}$ to exist, $x - y > 0 \Rightarrow x > y$



Answer: II & IV
(a)

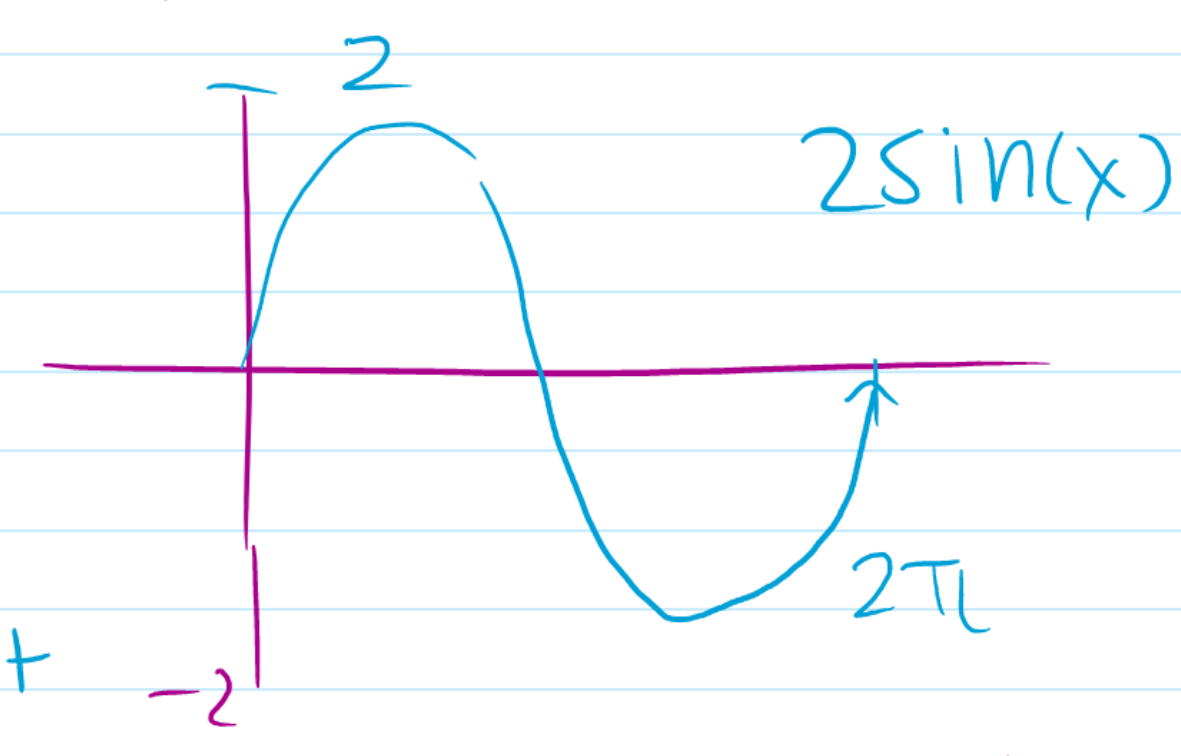
← use intersection

14. Draw a contour plot of $f(x,y) = \ln(y - 1 - 2\sin x)$.

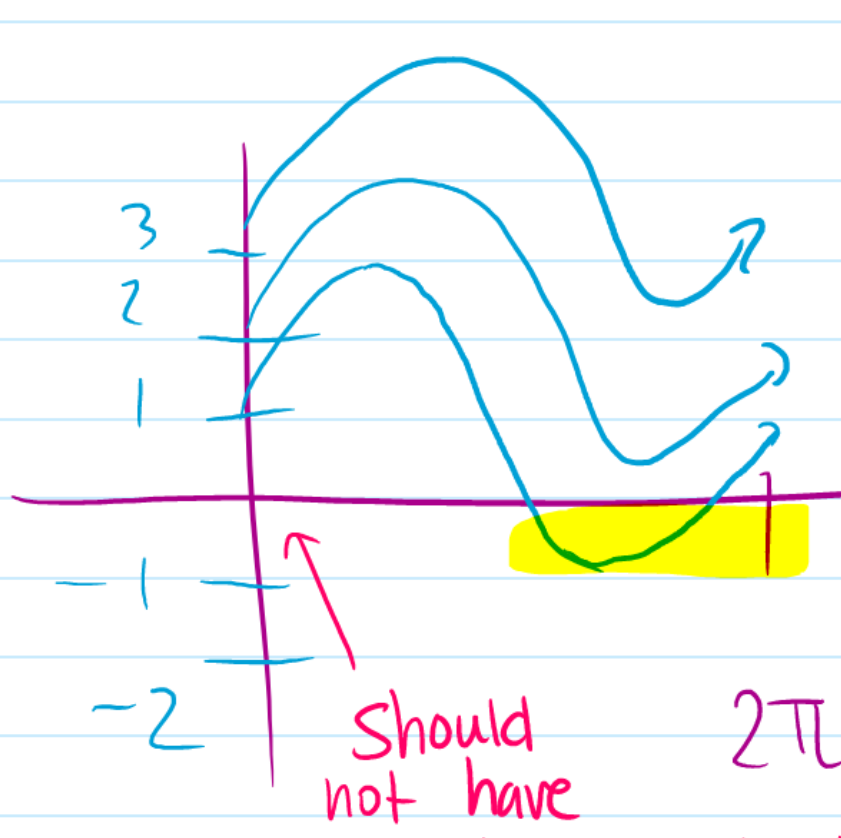
$k = \ln(y - 1 - 2\sin x)$

$e^k = y - 1 - 2\sin x$

$y = 2\sin(x) + \underbrace{e^k + 1}_{\text{constant}}$



You need to add values, ie, shift upwards by at least 1!



← like so.

allowed to have some negative

Should not have any starting pts here.

$2\pi \approx 2(3.14) = 6.24$

Closest to plot (e)

15. If $z = x^2 - xy + 3y^2$ and (x,y) changes from $(3,-1)$ to $(2.96, -0.95)$, find the value of the differential dz.
(a) $-\frac{73}{100}$ **(b)** $\frac{17}{100}$ **(c)** $\frac{73}{100}$ **(d)** $-\frac{17}{100}$ **(e)** $\frac{17}{10}$

$dz = f_x(x,y)dx + f_y(x,y)dy$

$f_x(x,y) = 2x - y$ $dx = 2.96 - 3 = -0.04$

$f_y(x,y) = -x + 6y$ $dy = -0.95 - (-1) = 0.05$

$dz = f_x(3,-1)(-0.04) + f_y(3,-1)(0.05)$

$= [2(3) - (-1)](-0.04) + [-3 + 6(-1)][0.05]$

$= [7][-0.04] + [-9][0.05]$

$= -0.28 - 0.45 = -0.73 = \frac{-73}{100}$ (a)

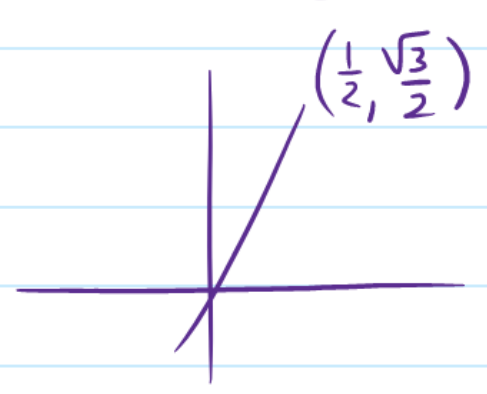
16. Find the directional derivative of $f(x,y) = x\sin(xy)$ at the point $(2,0)$ in the direction indicated by the angle $\pi/3$.
(a) $\frac{\sqrt{3}}{2} + \frac{1}{2}$ **(b)** $2\sqrt{3} + \frac{1}{2}$ **(c)** $2\sqrt{3}$ **(d)** $\frac{\sqrt{3}}{2}$ **(e)** $\frac{1}{2}$

Use: $D_uf(x,y) = f_x(x,y)\cos(\theta) + f_y(x,y)\sin(\theta)$

$f_x = \sin(xy) + x\cos(xy)y$ $f_x(2,0) = \sin(0) + 2\cos(0)(0) = 0$

$f_y = x\cos(xy)x$ $f_y(2,0) = 2^2\cos(0) = 4$

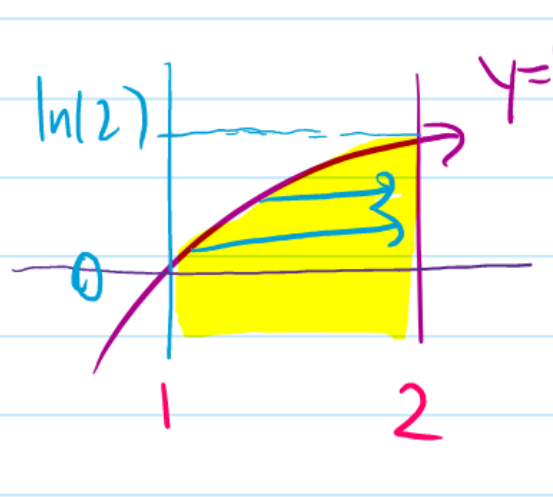
$D_uf(x,y) = 0\cos(\pi/3) + 4\sin(\pi/3) = 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$ (c)



17. Which of the below integrals is equal to

$$\int_1^2\int_0^{\ln x} f(x,y)dydx$$

(a) $\int_e^{e^2}\int_1^y f(x,y)dxdy$ **(b)** $\int_0^{\ln 2}\int_{e^y}^2 f(x,y)dxdy$ **(c)** $\int_0^{\ln x}\int_1^2 f(x,y)dxdy$
(d) $\int_0^1\int_{e^y}^1 f(x,y)dxdy$ **(e)** $\int_0^1\int_0^{\ln y} f(x,y)dxdy$



if you want w.r.t x first:
bottom is $y = \ln(x) \Rightarrow e^y = x$
highest value is $x = 2$

w.r.t y: you go from $\ln(1) = 0$ to $\ln(2)$

thus: $\int_0^{\ln(2)}\int_{e^y}^2 f(x,y)\,dx\,dy$ (b)