MATH 1AA3/1ZB3 Relevant Theorems

Steward Calculus - Sections 14 & 15

Anna Ly

Department of Mathematics & Statistics
McMaster University

April 4, 2025

Clairaut's Theorem

Clairaut's Theorem (MY ONE TRUE LOVE)

Suppose f is defined on a disk D that contains the point (a, b). If the function f_{xy} and f_{yx} are both continuous on D then,

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Why is this helpful?

- Usually they may only give you f_{xy} or f_{yx} only.
- Sometimes one is easier to compute than the other.

Equations 1

Equation of a Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Linear Approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Total Differential/Differential dz

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

Equations 2

Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y where $x = g(t)$ and $y = h(t)$ are both differentiable functions of $t/$ Then, z is a differentiable function of t and:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Equations 3

Implicit Differentiation (Case 1)

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0 \;\Rightarrow\; \frac{dy}{dx} = \frac{-F_x}{F_y}$$

Implicit Differentiation (Case 2)

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \;\Rightarrow\; \frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \;\&\; \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Directional Derivatives 1

Theorem

If f is a differentiable function of x and y then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b = \nabla f(x, y) \cdot \mathbf{u}$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis then write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and we have:

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos\theta + f_y(x, y) \sin\theta$$

Directional Derivatives 2

Gradient

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f:

$$\nabla f(x, y) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

If you had three variables, x, y, z then:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$$

Directional Derivatives 3 & More Equations

Maximum Value of the Directional Derivative

Suppose f is differentiable. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$.

Equation of the Tangent Plane

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Symmetric Equations of the Normal Line

$$\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_x(x_0, y_0, z_0)} = \frac{z - z_0}{F_x(x_0, y_0, z_0)}$$

Best Theorem for Multiple Integrals

Fubini's Theorem (MY ACTUAL ONE TRUE LOVE)

If f is continuous on the rectangle:

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

Then:

$$\int_a^b \int_c^d f(x, y) dydx = \int_c^d \int_a^b f(x, y) dxdy$$

Why is this helpful?

- It can be very easy to integrate when you switch sides.
- I've noticed most examples in your practice exam are so much easier if you reversed the order instead of doing it directly.