Tutorial 3

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HW3 Problem 4: use comparison test to determine convergence.

(i) $\sum_{n=1}^{\infty} \frac{2^{n+1}}{6^n+2}$ $\quad \frac{2^{n+1}}{6^n+2} \le \frac{2\cdot 2^n}{6^n} = 2\left(\frac{1}{3}\right)^n = \frac{2}{3}\left(\frac{1}{3}\right)^{n-1}$ Geometric Series.

since $1/3 = |r| \le 1$, $\sum_{n=1}^{\infty} \frac{2}{3}\left(\frac{1}{3}\right)^{n-1}$ converges

$\Rightarrow \sum_{n=1}^{\infty} \frac{2^{n+1}}{6^n+2}$ converges.

(ii) $\sum_{n=2}^{\infty} \frac{\sqrt{n^2-2}}{n^3+3}$ $\quad \sqrt{n^2-2} \le \sqrt{n^2} = n$

Recall: $\sum_{n=1}^{\infty} \frac{1}{n^p}$
if $p>1$: convergent
$p \le 1$: divergent

$\frac{\sqrt{n^2-2}}{n^3+3} \le \frac{n}{n^3+3}\frac{(1/n)}{(1/n)} = \frac{1}{n^2+3/n} \le \frac{1}{n^2}$

Hence $\sum_{n=2}^{\infty} \frac{1}{n^2}$ converges $\Rightarrow \sum_{n=2}^{\infty} \frac{\sqrt{n^2-2}}{n^3+3}$ converges.

(iii) $\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^4}$ $\quad \arctan(n) \le \frac{\pi}{2} \Rightarrow \frac{\arctan(n)}{n^4} \le \frac{\pi}{2}\frac{1}{n^4}$

since $\sum \frac{1}{n^4}$ (→ p series) converges so does $\sum \frac{\pi}{2}\frac{1}{n^4} = \frac{\pi}{2}\sum\frac{1}{n^4}$

So $\sum \frac{\arctan(n)}{n^4}$ converges

HW 3 Q5: convergent or divergent?

(i) $\sum_{n=1}^{\infty} \frac{8^n-2}{5^n}$ $\quad$ note: $6^n \le 8^n - 2 \Rightarrow \frac{6^n}{5^n} \le \frac{8^n-2}{5^n}$

note that $\left(\frac{6}{5}\right)^n = \frac{6}{5}\left(\frac{6}{5}\right)^{n-1}$ is a geometric series w/ $|r| = 6/5 \ge 1$

So, $\sum_{n=1}^{\infty} \frac{6}{5}\left(\frac{6}{5}\right)^{n-1}$ diverges $\Rightarrow \sum_{n=1}^{\infty} \frac{8^n-2}{5^n}$ diverges.

(ii) $\sum_{n=1}^{\infty} \frac{n^4+6n}{n^7+7n+7}$ $\quad \frac{n^4+6n}{n^7+7n+7} \le \frac{n^4+6n}{n^7} = \frac{n^4}{n^7} + \frac{6n}{n^7}$

$= \frac{1}{n^3} + \frac{6}{n^6} \le \frac{1}{n^2} + \frac{6}{n^2} = \frac{7}{n^2}$ $\quad \sum 7\frac{1}{n^2} = 7\sum\frac{1}{n^2}$ converges so

$\sum \frac{n^4+6n}{n^7+7n+7}$ converges.

(iii) $\sum_{n=1}^{\infty} \frac{\sqrt{n^7+3n}}{n^3+n}$ $\quad n^7 \le n^7+3n \Rightarrow \frac{\sqrt{n^7}}{n^3+n} \le \frac{\sqrt{n^7+3n}}{n^3+n}$

$\Rightarrow \frac{n^{7/2}}{n^3+n} \ge \frac{n^{7/2}}{n^3+n^3} = \frac{1}{2}n^{1/2}$ $\quad \sum_{n=1}^{\infty} \frac{1}{2}\sqrt{n}$ diverges

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n^7+3n}}{n^3+n}$ diverges.

HW3 Problem 6: more convergence vs. divergence

(i) $\sum_{n=1}^{\infty} \frac{3^{n+1}}{5^n}$ $\quad \frac{3^{n+1}}{5^n} = \frac{3^2\, 3^{n-1}}{5\; 5^{n-1}} = \frac{9}{5}\left(\frac{3}{5}\right)^{n-1}$

Again, geometric series.

This series converges b/c $3/5 = |r| < 1$

(ii) $\sum_{n=1}^{\infty} e^{1/n}$ $\quad$ note: $e^x > 1$ for all x, So,

$\sum_{n=1}^{\infty} 1$ diverges : $\sum_{n=1}^{\infty} e^{1/n}$ diverges.

(iii) $\sum_{n=1}^{\infty} \cos(1/n)$ $\quad$ Thm: if $\lim_{n\to\infty} a_n \ne 0$ or DNE then $\sum_{n=1}^{\infty} a_n$ is divergent.

DOESN'T MEAN $\lim_{n\to\infty} a_n = 0$ is CONVERGENT!

$\lim_{n\to\infty} \cos(1/n) = \cos(0)$
$= 1 \ne 0$ Hence $\sum_{n=1}^{\infty} \cos(1/n)$ is divergent.

HW3 Problem 7: $S_n = \frac{4n-4}{4n+2}$

a) find a_2 $\quad S_2 = a_1 + a_2$

$a_2 = S_2 - a_1 = S_2 - S_1 \Rightarrow a_2 = \frac{4(2)-4}{4(2)+2} - \frac{4-4}{4+2}$ (0)

$a_2 = \frac{4}{10} = \frac{2}{5}$

b) find $\sum_{n=1}^{\infty} a_n$ $\quad$ Recall: $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \sum_{i=1}^{\infty} a_i$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{4n-4}{4n+2}\frac{(1/n)}{(1/n)} = \lim_{n\to\infty} \frac{4-4/n}{4+2/n} = 1$

HW4 Problem 1: convergence or divergence?

(i) $\sum_{n=1}^{\infty} (-1)^{n-1} e^{-n/7}$ $\quad b_n = e^{-n/7} = \frac{1}{e^{n/7}}$

① check decreasing: $b_{n+1} \le b_n$

$\Leftrightarrow e^{-(n+1)/7} \le e^{-n/7} \Leftrightarrow \frac{1}{e^{(n+1)/7}} \le \frac{1}{e^{n/7}}$

$\Leftrightarrow e^{(n+1)/7} \ge e^{n/7}$ (true since e^x is incr.)

② $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{e^{n/7}} = 0$

Convergent by AST.

(ii) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3+4}$ difficult to compare: use calculus...

$f(x) = \frac{x^2}{x^3+4}$ $\quad f'(x) = \frac{2x(x^3+4) - x^2(3x^2)}{(x^3+4)^2}$

Warning: doesn't decrease for all x, but it will for $x \ge 2$. → it's fine!

$= \frac{2x^4+8x-3x^4}{(x^3+4)^2} = \frac{-x^4+8x}{(x^3+4)^2} = \frac{-x(x^3-8)}{(x^3+4)^2} < 0$

and $\lim_{n\to\infty} \frac{n^2}{n^3+4}\frac{(1/n^2)}{(1/n^2)} = \lim_{n\to\infty} \frac{1}{n+4/n^2} = 0.$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^3+4}$ converges by AST.

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+4}}$ $\quad b_{n+1} \le b_n \Leftrightarrow \frac{1}{\sqrt{n+5}} \le \frac{1}{\sqrt{n+4}}$

$\Leftrightarrow \sqrt{n+5} \ge \sqrt{n+4} \Leftrightarrow n+5 \ge n+4 \Leftrightarrow 5 \ge 4$

So it increases and:

$\lim_{n\to\infty} \frac{1}{\sqrt{n+4}} = 0$ $\quad$ thus it converges by AST.