Tutorial 4

January 31, 2025 7:01 PM

HMWK 4 Problem 3: which are conditionally convergent?

(i) $\sum_{n=1}^{\infty} (-1)^n \frac{\arctan(n)}{n^2}$ $\quad |a_n| = \frac{\arctan(n)}{n^2} \le \frac{\pi}{2}\frac{1}{n^2}$ → p-series $p=2>1$ ↳ converges

since $|a_n|$ converges ⇒ this converges absolutely!

(ii) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{\sqrt{n^3+2}}$ $\quad |a_n| = \frac{n}{\sqrt{n^3+2}}$

$\frac{n}{\sqrt{n^3+2}} \ge \frac{n}{\sqrt{n^3+2n^3}} = \frac{n}{\sqrt{3n^3}} = \frac{n}{\sqrt{3}\,n^{3/2}} = \frac{1}{\sqrt{3}\,n^{1/2}}$ → p-series $p=\frac{1}{2}\le 1$ ↳ diverges

Hence $|a_n|$ diverges. Does a_n converge? Let's try AST...

$b_n = \frac{n}{\sqrt{n^3+2}}$ $\quad f(x) = \frac{x}{(x^3+2)^{1/2}}$ $\quad f'(x) = \frac{(x^3+2)^{1/2} - x(\frac{1}{2})(x^3+2)^{-1/2}(3x^2)}{(x^3+2)}$

$= \frac{(x^3+2)^{-1/2}\left((x^3+2) - \frac{3}{2}x^3\right)}{(x^3+2)}$

So, b_n decreases!

$= \frac{-\frac{x^3}{2}+2}{(x^3+2)^{1/2}}$ $\quad$ note: $-\frac{x^3}{2}+2<0$

And, $\lim_{n\to\infty} b_n = 0$. $\quad \iff 2 < \frac{x^3}{2} \iff 4 < x^3$

By AST, $\sum a_n$ converges. $\quad \iff (4)^{1/3} < x$ decreases when $x > (4)^{1/3}$.

HENCE, this is conditionally convergent.

(iii) $\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ $\quad |a_n| = \frac{1}{\ln(n)}$ $\quad$ note: $n > \ln(n)$, $\frac{1}{n} < \frac{1}{\ln(n)}$, $p=1$ diverges

so $\sum |a_n|$ doesn't converge.

Let's see if $\sum a_n$ converges by AST,

1. $b_n \ge b_{n+1} \iff \frac{1}{\ln(n)} \ge \frac{1}{\ln(n+1)} \iff \ln(n+1) \ge \ln(n)$
$\ln(\cdot)$ is an incr function.

2. $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$ $\quad$ Hence by AST $\sum a_n$ converges.

Thus $\sum a_n$ is conditionally convergent.

HW4 Problem 4: Ratio Test for convergence!

(i) $\sum_{n=1}^{\infty} \frac{9^{n-1}}{(n+2)!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{9^n}{(n+3)!} \div \frac{9^{n-1}}{(n+2)!}\right| = \frac{9^n}{9^{n-1}} \frac{(n+2)!}{(n+3)!}$

$= \frac{9}{(n+3)}$ $\quad \lim_{n\to\infty} \frac{9}{(n+3)} = 0 < 1 \Rightarrow$ series converges absolutely.

(ii) $\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{3^{n+1}}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}(n+1)^4}{3^{n+2}} \times \frac{3^{n+1}}{(-1)^n n^4}\right| = \frac{(n+1)^4}{n^4}\frac{1}{3}$

$\lim_{n\to\infty} \frac{(n+1)^4}{3n^4} = \frac{1}{3} < 1 \Rightarrow$ series converges absolutely.

$(n+1)^4 = n^4+3n^3+3n^2+1$ $\quad$ (binomial thm expansion)

1
1 2 1
1 3 3 1

(iii) $\sum_{n=2}^{\infty} \frac{n^5}{n!\ln(n)}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^5}{(n+1)!\ln(n+1)} \frac{n!\ln(n)}{n^5}\right| = \frac{(n+1)^5}{n^5}\frac{n!}{(n+1)!}\frac{\ln(n)}{\ln(n+1)}$

$= \frac{(n+1)^5}{n^5}\frac{1}{n+1}\frac{\ln(n)}{\ln(n+1)} = \frac{(n+1)^4}{n^5}\frac{\ln(n)}{\ln(n+1)}$

$\lim_{n\to\infty} \frac{(n+1)^4}{n^5}\frac{\ln(n)}{\ln(n+1)} = 0 < 1 \Rightarrow$ converges absolutely.

HW4 Problem 5 simplify $\frac{a_{n+1}}{a_n}$ given $\sum_{n=1}^{\infty} \frac{2^n(3n)!}{1\cdot 5\cdot 9\cdots(4n+1)}$

$a_{n+1} = \frac{2^{n+1}(3n+3)!}{1\cdot 5\cdot 9\cdots(4n+1)(4n+5)}$ $\quad$ +4 +4

$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}(3n+3)!}{\cancel{2^n}(3n)!} \times \frac{\cancel{1\cdot 5\cdot 9\cdots(4n+1)}}{\cancel{1\cdot 5\cdot 9\cdots(4n+1)}(4n+5)} = \frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{\cancel{3n!}}\frac{2}{4n+5}$

$= \frac{2(3n+3)(3n+2)(3n+1)}{4n+5}$

HW4 Problem 6 find interval of convergence:

$\sum_{n=1}^{\infty} \frac{(x-9)^n}{6^n\sqrt{n}}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-9)^{n+1}}{6^{n+1}\sqrt{n+1}} \times \frac{6^n\sqrt{n}}{(x-9)^n}\right| = \left|\frac{(x-9)}{6}\frac{\sqrt{n}}{\sqrt{n+1}}\right|$

$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}\frac{|x-9|}{6} = \frac{|x-9|}{6} < 1 \Rightarrow |x-9| < 6 \Rightarrow -6 < x-9 < 6$
for convergence $\quad \Rightarrow 3 < x < 15$

Now, if $x=3$ then: $\frac{(-6)^n}{6^n\sqrt{n}} = \frac{(-1)^n}{\sqrt{n}}$ use AST. $b_n = \frac{1}{\sqrt{n}}$

(i) $\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} \iff \sqrt{n+1} > \sqrt{n} \iff n+1 > n$ ✓

(ii) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ converges.

If $x=15$ then: $\frac{6^n}{6^n\sqrt{n}} = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ p-series $p=\frac{1}{2}\le 1 \Rightarrow$ diverges.

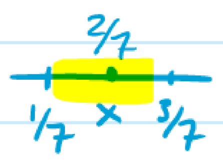
Interval for convergence: $[3, 15)$.

HW 4 Problem 7 find radius of convergence.

$\sum_{n=1}^{\infty} \frac{(7x-2)^n}{n^2}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(7x-2)^{n+1}}{(n+1)^2} \times \frac{n^2}{(7x-2)^n}\right| = |7x-2|\frac{n^2}{(n+1)^2}$

$\lim_{n\to\infty} |7x-2|\frac{n^2}{(n+1)^2} = |7x-2| < 1 \Rightarrow -1 < 7x-2 < 1$

$\Rightarrow 1 \le 7x \le 3 \Rightarrow \frac{1}{7} \le x \le \frac{3}{7}$



has radius $\frac{1}{7}$.