MATH 1AA3/1ZB3 Term Test 2 Review

(Mostly Power & Taylor Series)

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Overview

1. Power Series
2. Taylor Series
3. Area of a Surface of Revolution
4. Differential Equations
5. Parametric Equations

Power Series

- In mathematics, "power" is referring to the exponent.
- Naturally, if you hear "power series" you should also be thinking of exponents.
- The power series is just denoted as a sum of exponents, i.e.,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

- This series converges for when $|x| < 1$.
- The choice of x is arbitrary, which is why we like to substitute it for many examples.

Power Series Example 1

$$\text{Recall:} \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Now consider the following function, where $a, b, c, d \in \mathbb{N}$. How can you write this as a power series?

$$f(x) = \frac{ax^b}{c + dx^2}$$

First, ignore the numerator. Then, try to divide by the additive term so you have a form as a 1. Then, try to find what your new "x" is to use as a power series representation!

Power Series Example 1

$$\begin{aligned}
\frac{ax^b}{c + dx^2} &= ax^b \frac{1}{c + dx^2} \frac{1/c}{1/c} \\
&= ax^b 1/c \frac{1}{1 + (dx^2/c)} \\
&= \frac{ax^b}{c} \frac{1}{(1 - (-dx^2/c))} \\
&= \frac{ax^b}{c} \sum_{n=0}^{\infty} \left(\frac{-dx^2}{c} \right)^n
\end{aligned}$$

Interesting Derivatives

$$\text{Recall:} \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

Now take the derivatives on both sides. What do we get?

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{-1}{(1-x)^2(-1)} = \frac{1}{(1-x)^2}$$

$$\frac{d}{dx}\left(1 + x^1 + x^2 + x^3 + ...\right) = 0 + 1 + 2x + 3x^2 + ... = \sum_{n=1}^{\infty} nx^{n-1}$$

You could technically start at $n = 0$; it's just that adding 0 is redundant. But sometimes we'll want to use it.

We can use this trick to find the sum of certain series... It can get tricky though!

Power Series Example 2

Find the sum of $\sum_{n=1}^{\infty} n^2\delta^n$, for $\delta \in \mathbb{R}$.
Previously we showed that (using the $n = 0$ version):

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

Multiply both sides by x.

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n}$$

Now we can take the derivative again...

Power Series Example 2

$$\frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

$$\frac{(1+x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n-1}$$

And again, multiply both sides by x.

$$\frac{x(1+x)}{(1-x)^3} = \sum_{n=1}^{\infty} n^2 x^{n}$$

Power Series Example 2

Hence, for $\sum_{n=1}^{\infty} n^2\delta^n$ you would then just plug in the value for $x = \delta$:

$$\sum_{n=1}^{\infty} n^2\delta^n = \frac{\delta(1+\delta)}{(1-\delta)^3}$$

I.e., suppose $\delta = 1/2$. Then we have:

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{1/2(1+(1/2))}{(1-(1/2))^3} = \frac{1/2(3/2)}{(1/2)^3} = \frac{3/2}{(1/2)^2} = 6$$

The Connection to Natural Logarithm

Recall that:

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

And with the $\mathcal{MAGICAL}$ of chain rule, for $a, b \in \mathbb{R}$:

$$\frac{d}{dx}(\ln(a + bx)) = \frac{b}{a + bx}$$

There is of course a connection with $\ln(\cdot)$ and the power series!

The Connection to Natural Logarithm

Natural Question

What's the point of using power series!?

Brief Answer

Polynomials are nicer than dealing with logarithms, exponentials, etc.
For example, it's very easy to take the derivative or integral of them.
Sometimes you don't care about exact number but an approximation; power series are great for this.
We'll talk about it more when we look at remainder theorem of Taylor Series.

Power Series Example 3

Let $a, b \in \mathbb{N}$. Represent the following as a power series:

$$\ln(a - bx^2)$$

Again, we first take the derivative:

$$\frac{d}{dx}\left(\ln(a - bx^2)\right) = \frac{-2bx}{a - bx^2}$$

Now we want to re-represent the series as a power series.

$$-2bx\frac{1}{a - bx^2}\frac{1/a}{1/a} = \frac{-2bx}{a}\frac{1}{(1 - (bx^2/a))} = \frac{-2bx}{a}\sum_{n=0}^{\infty}\left(\frac{bx^2}{a}\right)^n = -2\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^{n+1}x^{2n+1}$$

Power Series Example 3

So currently we have:

$$\begin{aligned}
\ln(a - bx^2) &= \int \frac{d}{dx}\left(\ln(a - bx^2)\right) dx \\
&= \int -2\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^{n+1} x^{2n+1} dx \\
&= -2\int \left\{\left(\frac{b}{a}\right)x^1 + \left(\frac{b}{a}\right)^2 x^3 + \left(\frac{b}{a}\right)^3 x^5 + \ldots\right\} dx \\
&= -2\left[\left(\frac{b}{a}\right)\frac{x^2}{2} + \left(\frac{b}{a}\right)^2 \frac{x^4}{4} + \left(\frac{b}{a}\right)^3 \frac{x^6}{6} + \ldots\right] \\
&= -2\sum_{n=0}^{\infty}\left(\frac{b}{a}\right)^{n+1} \frac{x^{2n+2}}{(2n+2)}
\end{aligned}$$

Taylor Series

Similar to power series, the idea is to re-write functions as polynomials and their derivatives.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f^{(1)}(a)}{1!}(x-a)^1 + \frac{f^{(2)}(a)}{2!}(x-a)^2 + ...$$

The partial sums, i.e., $\sum_{n=0}^{N}$ often serve as good approximations!

$$T_n(x) = \sum_{i=1}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Maclaurin Series is the special case where $a = 0$.

Taylor's Inequality

Here we are going to motivate WHY we use Taylor series often...

Taylor's Inequality

If $|f^{(n+1)}| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality:

$$|f(x) - T_n(x)| = |R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \quad |x-a| \leq d$$

Essentially, this theorem tells us that we can calculate an upper bound of the remainder by looking at the absolute value of the $(n+1)th$ derivative multiplied by some other terms.

Taylor's Inequality

Natural Question

Why is this theorem useful?

Brief Answer

In real life we don't need exact answers.
For example, suppose I told you the probability of passing a test is 72.34961293885%.
You probably only wanted to hear me say 72.3%, which is the approximation.
Similarly, going from $n = 0$ to ∞ can be overkill. It's probably sufficient to just estimate the first n terms, where we can choose n depending on the upper bound. I.e., up to 0.001 or 0.0001 are common thresholds.

Taylor's Inequality - Common Pitfalls

Pitfall 1

Some questions will tell you to find an upper bound while giving you the actual value of $T_n(x)$. I.e., they could say $T_4(x) = \frac{4!}{x^4}$.
Note that the inequality is actually more focused towards the values $|f^{(n+1)}| \leq M$ and $\frac{M}{(n+1)!}|x-a|^{n+1}$. So, the given function of $T_n(x)$ is actually a bit of a red-herring.
You generally only need to know the nth value, which above was just $n = 4$.

Pitfall 2

You might've heard that $M = \sup\{|f^{(n+1)}(x)|\}$ (the least upper bound).
"least" and "upper" sound like a contradiction, but there's a reason we use this.
Think of $f(x) = -x^2$. The least upper bound is $0 \geq -x^2$.
However, I can also claim $100 \geq -x^2, 391 \geq -x^2, 509208402 \geq -x^2$...
Basically, larger bounds than the least upper-bound provides less information! We really care about trying to find a bound for the remainder $|R_n(x)| = |f(x) - T_n(x)|$.

Taylor's Inequality Example

Question

Anna really wanted to compute $f(x) = \ln(1+2x)$ centered at $a = 1$ for $1/2 \leq x \leq 3/2$. However, the computer exploded. Now Anna needs to do this by hand. How good is the approximation if we used $T_3(x) = \ln(3) + \frac{2}{3}(x-1) - \frac{2}{9}(x-1)^2 + \frac{8}{81}(x-1)^3$?

REMEMBER: the formula for $T_3(x)$ is a red-herring. What's important is to locate that $n = 3$. Hence, the first step now to compute $f^{(4)}(x)$:

$$f^{(0)}(x) = \ln(1+2x), \quad f^{(1)}(x) = \frac{2}{1+2x}, \quad f^{(2)}(x) = \frac{-2^2}{(1+2x)^2}$$

$$f^{(3)}(x) = \frac{2^3(2!)}{(1+2x)^3}, \quad f^{(4)}(x) = \frac{-2^4(3!)}{(1+2x)^4}$$

So now we have that:

$$|f^4(x)| = \frac{2^4(3!)}{(1+2x)^4}$$

Taylor's Inequality Example

$$|f^4(x)| = \frac{2^4(3!)}{(1+2x)^4}$$

Now to find M, the least upper-bound, AND we have that $\frac{1}{(1+2x)^4}$ is monotone for $x \geq 0$ and our domain is $1/2 \leq x \leq 3/2$, we'll need to check for end points $x = 1/2$ and $x = 3/2$. Compare:

$$x = 1/2 : \frac{1}{(1+2(1/2))^4} = \frac{1}{2^4} \quad \text{v.s.} \quad x = 3/2 : \frac{1}{(1+2(3/2)^4)} = \frac{1}{4^4}$$

Clearly, $x = 1/2$ gives us a larger value since $\frac{1}{2^4} > \frac{1}{4^4}$.
So our choice of M occurs when $x = 1/2$, leading us with:

$$M = |f^4(1/2)| = \frac{2^4(3!)}{(1+2(1/2))^4} = 3! = 6$$

Taylor's Inequality Example

Now we want to use:

$$|f(x) - T_3(x)| = |R_3(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} = \frac{6}{4!}|x-1|^4$$

Similarly to before, we want to pick x such that we get the highest possible bound. $|\cdot|$ isn't necessarily a monotone function, however, $|\cdot|$ is monotone increasing to the right of the center and monotone decreasing to the left of the center.
For example, if you had $|a - bx|$ then it is increasing to the right of b and decreasing to the left of b.
So it is sufficient to just check the endpoints:

$$x = 1/2 : |(1/2) - 1|^4 = |1/2|^4 = 1/16, \quad x = 3/2 : |(3/2) - 1|^4 = |1/2|^4 = 1/16,$$

Both would give you the same answer. Hence...

Taylor's Inequality Example

Now we want to use:

$$|R_3(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} = \frac{6}{4!}|x-1|^4 \leq \frac{6}{4!}\frac{1}{16} = \frac{1}{64} \approx 0.016$$

Is this good enough of an estimate? Really depends on the field! :)

Binomial Series

To understand binomial series, or at least to understand how to simplify, you need to understand binomial coefficients.

Binomial Coefficients

$$\binom{k}{n} = \frac{k(k-1)(k-2)...(k-n+1)}{n!}$$

Aside: in combinatorics, the notation of $\binom{n}{k}$ is more common...

So what does this expression look like? Let's do some test points...

$$\binom{23}{4} = \frac{(23)(22)(21)(20)}{4!}, \quad \binom{13}{6} = \frac{(13)(12)(11)(10)(9)(8)}{6!}$$

Observation: the number of multiplicative terms in the numerator is equal to the nth value.

Binomial Series

The Binomial Series

If k is any real number and $|x| < 1$ then:

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

This is similar to power series, except now we focus on expressions of the form $(1+x)^k$ instead of just $\frac{1}{1-x}$.

Binomial Series Example

Now consider the following function, where $a, b, c \in \mathbb{N}$. How can you write this as a binomial series?

$$f(x) = (a + bx)^c$$

Note that:

$$(a + bx)^c = \left(\frac{a}{a}(a + bx)\right)^c = a^c \left(1 + \frac{b}{a}x\right)^c = a^c \left[\sum_{n=0}^{\infty} \binom{c}{n} \left(\frac{b}{a}x\right)^n\right]$$

Now of course, since the choice of a, b, c were arbitrary, there isn't a nice pattern for expansion. However, with integers you may be able to simplify the process more by observing patterns and so on...

Important Maclaurin Series

In a nutshell: you'll need to memorize them (both ways).

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad (1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

I didn't include their radius of convergence; you should remember them too!

What are some patterns you observe to make it easier to memorize?

Maclaurin Series Example

Can you identify the expression for the series? Again, let $a, b, c, d \in \mathbb{N}$.

Examples

1. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n!a^n}$, 2. $\displaystyle(cd)\sum_{n=0}^{\infty}(-1)^n \frac{(cd)^{2n}}{2n+1}$, 3. $\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{(b)^{n+1}}{n+1}$

Maclaurin Series Example

Note that:

$$\sum_{n=0}^{\infty} \frac{1}{n!a^n} = \sum_{n=0}^{\infty} \frac{(1/a)^n}{n!} = e^{1/a}$$

$$(cd)\sum_{n=0}^{\infty}(-1)^n \frac{(cd)^{2n}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{(cd)^{2n+1}}{2n+1} = \tan^{-1}(cd)$$

The last one is tricky.

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$$

Thus,

$$\sum_{n=0}^{\infty}(-1)^n \frac{(b)^{n+1}}{n+1} = \ln(1+b)$$

Area of a Surface of Revolution Formulas

This section is mostly plug-in and practicing your algebra.

Surface area rotating about the $x-$axis:

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx$$

Surface area rotating about the y-axis

:

$$S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx, \quad \text{or} \quad S = \int_a^b 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy,$$

Advice for rotating about the y-axis: see if the multiple choice answers uses dx or dy.

Differential Equations

A differential equation is an equation that depends on a function and its derivatives.
Examples:

$$f(x) + f^{(1)}(x) + f^{(2)}(x) = 4x, \quad f(x) + f^{(345)} = \frac{1}{\ln(x)}, \quad \text{etc...}$$

Unless they're separable equations, they're generally extremely hard to derive!
Lots of applications in science and engineering!
Differential equations are a "late-game" thing for statisticians (i.e., advanced stochastic calculus) so your TA, who is studying statistics, isn't an expert for applications.

Exponential Growth and Decay

These are the easiest differential equations.

Law of Natural Growth

$f'(t) = kf(t)$. People instead often write:

$$\frac{dy}{dt} = kt \;\Rightarrow\; y(t) = y(0)e^{kt}$$

Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_s) \;\Rightarrow\; T(t) = T(0)e^{kt} + T_s$$

Warning: sometimes you won't have the initial condition $y(0)$ or $T(0)$; it actually may vary depending on what they give you. So it is good to derive the equation yourself.

Separable Equations

Everyone's favourite! Solve for y using this method:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \Rightarrow \int h(y)dy = \int g(x)dx$$

Mixing Problems

These often will result in a separable equation:

$$\frac{dy}{dt} = (\text{rate in}) - (\text{rate out})$$

Usually, either (rate in) or (rate out) will be in terms of y and the other will be a constant.

Orthogonal Trajectory Method

Best explained through an example. Given $a \geq 2, b \geq 2 \in \mathbb{N}$ and $a < b$...

Orthogonal Trajectory Example

Find the orthogonal trajectories of the family of curves $y^a = kx^b$.

Take derivatives with respect to x on both sides:

$$ay^{a-1}\frac{dy}{dx} = bkx^{b-1}$$

Solve for $\frac{dy}{dx}$ and replace $k = \frac{y^a}{x^b}$

$$\frac{dy}{dx} = \frac{bkx^{b-1}}{ay^{a-1}} = \frac{b\frac{y^a}{x^b}x^{b-1}}{ay^{a-1}} = \frac{b}{a}\frac{y}{x}$$

Orthogonal Trajectory Method

Now we want to take the reciprocal:

$$\frac{b}{a}\frac{y}{x} \quad \Rightarrow \quad -\frac{a}{b}\frac{x}{y}$$

Hence,

$$\frac{dy}{dx} \overset{\text{set}}{=} -\frac{a}{b}\frac{x}{y} \quad \Rightarrow \quad \int bydy = \int -(ax)\,dx = \frac{by^2}{2} = \frac{-ax^2}{2} + C$$

$$\Rightarrow \quad \frac{by^2}{2} + \frac{ax^2}{2} = C$$

Note: the choice of a, b were arbitrary... I may have given you a formula you can just use...

Linear Differential Equations

People don't like these as much. Probably the situation where you won't have a separable equation.

Linear Differential Equations

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Strategy is to multiply both sides by the **integrating factor** $I(x) = \exp\{\int P(x)dx\}$ and integrate both sides.
Again, it's better to explain through an example.

Linear Differential Equation Example

Solve the differential equation

$$\frac{10}{x}y' + 20y = 10x^2e^{x^2}$$

First, we need y' to be alone. So divide all by 10 and multiply by x:

$$y' + 2xy = x^3e^{x^2}$$

Here, $P(x) = 2x$. I like to start small. (Also, the constant doesn't matter here.)

$$\int P(x)dx = \int 2xdx = x^2 \;\Rightarrow\; e^{\int P(x)dx} = e^{x^2}$$

Multiply everything by the integrating factor now and $\mathbb{INTEGRATE}$.

Linear Differential Equation Example

$$\int (e^{x^2}y' + e^{x^2}2xy)dx = \int e^{x^2}x^3e^{x^2}dx$$

Note that:

$$(e^{x^2}y)' = 2xe^{x^2}y + e^{x^2}y' \;\; \Rightarrow \;\; \int (e^{x^2}y' + e^{x^2}2xy)dx = e^{x^2}y$$

Furthermore,

$$\int e^{x^2}x^3e^{x^2}dx = \int e^{2x^2}x^3dx$$

Okay, this is going to be a tough integral. This is left for you to do as an exercise!

Linear Differential Equation Example

$$\int e^{2x^2} x^3 dx = \frac{(2x^2 - 1)e^{2x^2}}{8} + C$$

Therefore,

$$e^{x^2} y = \frac{(2x^2 - 1)e^{2x^2}}{8} + C \quad \Rightarrow \quad y = \frac{(2x^2 - 1)e^{x^2}}{8} + \frac{C}{e^{x^2}}$$

And now you plug in with the initial conditions and other things they may want to torture you with.

Parametric Equations

Suppose x and y are now functions of a third variable t.

$$x = f(t), \quad y = g(t)$$

Turning this back into Cartesian form is not bad!

Converting to Cartesian Example

$$x = t^2, \quad y = t^3$$

Note that $\sqrt{x} = t$ and therefore $y = (\sqrt{x})^3 \;\Rightarrow\; y = x^{3/2}$ which is the Cartesian form. Very nice.

Now, graphing these equations by hand really amounts to brute force. You get quicker if you remember patterns.

Finding Slopes of Tangent Lines

Best explained through an example.

Find an equation of the tangent line to the curve.

$$x = \sqrt{t+4}, \quad y = 1/(t+4), \quad \text{point: } (2, 1/4)$$

Note that:

$$\frac{dy}{dt} = \frac{-1}{(t+4)^2}, \quad \frac{dx}{dt} = \frac{1}{2\sqrt{t+4}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2\sqrt{t+4}}{(t+4)^2} = \frac{-2}{(t+4)^{3/2}}$$

Now we need to find a value for t using the coordinate point of x. Note that,

$$2 = \sqrt{t+4} \;\Rightarrow\; t = 0$$

Hence the slope is indeed $\frac{-2}{(4)^{3/2}} = -1/4$.

Finding Slopes of Tangent Lines

Using the classical $y = mx + b$:

$$\frac{1}{4} = \frac{-1}{4}(2) + b \;\Rightarrow\; b = \frac{3}{4}$$

And therefore the equation of the tangent line is:

$$y = \frac{-1}{4}x + \frac{3}{4}$$

How splendid.

Computing Area and Surface Areas

Computing Area Under a Curve

Again letting $x = f(t), \; y = g(t)$ for $\alpha \leq t \leq \beta$, use the area equation:

$$A = \int_{\alpha}^{\beta} g(t) f'(t) dt$$

Computing Arc Length Under a Curve

Letting curve C being defined by $x = f(t), \; y = g(t)$, $\alpha \leq t \leq \beta$, then the length of C is:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

WARNING: $f'(t)$ and $g'(t)$ must be continuous functions on $[\alpha, \beta]$. (Although if this doesn't happen, it's difficult to compute.)

The End

Yeah this was a lot. Hope you found this review useful. Best of luck with your test!