Question 3 (11.15 from the textbook)

1. Derive the following identity:

$$SSE = S_{yy} - \hat{\beta}_1 S_{xy}$$

This provides an easier computational method of finding the SSE.

2. Use the computational formula for SSE derived in part (a) to prove that $SSE \leq S_{yy}$. [**Hint**: $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}}$]

1. From the definition, we know $SSE = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$

As we know, $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$ and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

Then $SSE = \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta} x_i)^2 = \sum_{i=1}^{n} (y_i - \bar{y} + \hat{\beta}_1 \bar{x} - \hat{\beta}_1 x_i)^2$

$$= \sum_{i=1}^{n} (y_i - \bar{y} - \hat{\beta}_1 (x_i - \bar{x}))^2$$

$$= \sum_{i=1}^{n} (y_i - \bar{y})^2 - 2 \sum_{i=1}^{n} (y_i - \bar{y}) \cdot \hat{\beta}_1 (x_i - \bar{x}) + \sum_{i=1}^{n} \hat{\beta}_1^2 (x_i - \bar{x})^2$$

$$= \sum_{i=1}^{n} (y_i - \bar{y})^2 - 2\hat{\beta}_1 \sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x}) + \hat{\beta}_1^2 \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$= S_{yy} - 2\hat{\beta}_1 S_{xy} + \hat{\beta}_1^2 S_{xx}$$

$$= S_{yy} - 2\hat{\beta}_1 S_{xy} + \hat{\beta}_1^2 \cdot \underbrace{\frac{S_{xy}}{\beta_1}}$$

Since $\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} \Rightarrow S_{xx} = \frac{S_{xy}}{\hat{\beta}_1}$

$$= S_{yy} - 2\hat{\beta}_1 S_{xy} + \hat{\beta}_1 S_{xy} = S_{yy} - \hat{\beta}_1 S_{xy}$$

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2. $SSE = S_{yy} - \hat{\beta}_1 S_{xy}$

$$S_{yy} - SSE = \hat{\beta}_1 \cdot S_{xy} = \frac{S_{xy}}{S_{xx}} \cdot S_{xy} = \frac{S_{xy}^2}{S_{xx}}$$

$\because$ $S_{xx} > 0$ and $S_{xy}^2 \geqslant 0$.

$\therefore$ $S_{yy} - SSE \geqslant 0$

$\therefore$ $SSE \leq S_{yy}$