Question 4 (11.21 from the textbook)

Suppose that $Y_1, Y_2, ..., Y_n$ are independent normal random variables with $\mathbb{E}(Y_i) = \beta_0 + \beta_1 x_i$ and $\mathbb{V}(Y_i) = \sigma^2$, for $i = 1, 2, ..., n$. Find $Cov(\hat{\beta}_0, \hat{\beta}_1)$.
Then, prove that if $\sum_{i=1}^{n} x_i = 0$ then $\hat{\beta}_0, \hat{\beta}_1$ are independent.

As we know $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$

Then $Cov(\hat{\beta}_0, \hat{\beta}_1) = Cov(\bar{y} - \hat{\beta}_1 \bar{x}, \hat{\beta}_1)$

$= Cov(\bar{y}, \hat{\beta}_1) - Cov(\hat{\beta}_1 \bar{x}, \hat{\beta}_1)$

$= 0 - \bar{x}\, Cov(\hat{\beta}_1, \hat{\beta}_1)$

$= -\bar{x}\, Var(\hat{\beta}_1) = -\bar{x} \cdot \frac{\sigma^2}{S_{xx}}$

if $\sum_{i=1}^{n} x_i = 0$, then $\bar{x} = 0$, which means $Cov(\hat{\beta}_0, \hat{\beta}_1) = 0$.

therefore $\hat{\beta}_0, \hat{\beta}_1$ are independent.