STA260 Tutorial 10 Question 1

Question 1

Let $Y_1, ..., Y_8$ be a random sample from the probability density function given by:

$$f(x|\beta) = \begin{cases} \frac{3}{\beta}e^{-y^3/\beta} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the Rejection Region (RR) for the Uniformly Most Powerful (UMP) test of:

$$H_0 : \beta = 2 \quad \text{v.s.} \quad H_a : \beta > 2$$

with significance level $\alpha = 0.05$?

Hint:

$$\sum_{i=1}^{8} Y_i^3 \sim Gamma(8, 2)$$

Use NP Lemma: $L(y_1, \dots y_8 \mid \beta) = \frac{3}{\beta} e^{-y_1^3/\beta} \times \dots \times \frac{3}{\beta} e^{-y_8^3/\beta}$

$= (3/\beta)^8 e^{-1/\beta \sum_{i=1}^{8} y_i^3}$ hence:

$$\frac{L(\beta_0)}{L(\beta_a)} = \frac{\prod_{i=1}^{8} f(y_i|\beta_0)}{\prod_{i=1}^{8} f(y_i|\beta_a)} = \frac{(3/2)^8 e^{-1/2 \sum_{i=1}^{8} y_i^3}}{(3/\beta_a)^8 e^{-1/\beta_a \sum_{i=1}^{8} y_i^3}} = (\beta_a/2)^8 e^{-\sum_{i=1}^{8} y_i^3 \left(\frac{1}{2} - \frac{1}{\beta_a}\right)} \overset{\text{set}}{<} k$$

Since $\beta_a > 0$, $(\beta_a/2)^8 > 0$ and thus moving it over will not change the sign. Thus,

$$\frac{\cancel{\left(\frac{\beta_a}{2}\right)^8} e^{-\sum_{i=1}^{8} y_i^3 \left(\frac{1}{2} - \frac{1}{\beta_a}\right)}}{\cancel{\left(\frac{\beta_a}{2}\right)^8}} < \frac{k}{\left(\frac{\beta_a}{2}\right)^8} = k_1$$

Similarly, $\ln\left(e^{-\sum_{i=1}^{8} y_i^3 \left(\frac{1}{2} - \frac{1}{\beta_a}\right)}\right) < \ln(k_1) = k_2$

$$\frac{-\sum_{i=1}^{8} y_i^3 \cancel{\left(\frac{1}{2} - \frac{1}{\beta_a}\right)}}{\cancel{\left(\frac{1}{2} - \frac{1}{\beta_a}\right)}} < \frac{k_2}{\left(\frac{1}{2} - \frac{1}{\beta_a}\right)} = k_3$$

note: $\beta_a > 2$ then $(1/2 - 1/\beta_a) > 0$

$$-\sum_{i=1}^{8} y_i^3 < k_3 \Rightarrow \sum_{i=1}^{8} y_i^3 > -k_3 = k^*$$

Now, from the hint: $\sum_{i=1}^{8} Y_i^3 \sim Gamma(8, \beta)$

and if $\beta = 2$ then $Gamma(8, 2) = \chi^2_{(16)}$

Recall: $\chi^2_{(n)} = Gamma(n/2, 2)$

Hence, $\alpha = P\left(\sum_{i=1}^{8} Y_i^3 > k^* \mid \beta_0 = 2\right)$

here, $k^* = \chi^2_{df=16,\, p=0.05} = 26.296$

hence $RR = \left\{\sum_{i=1}^{8} Y_i^3 > 26.296\right\}$