STA260 Tutorial 10 Question 2

**Question 2**

Let $X_1, X_2, \ldots, X_n$ be independent random variables such that each $X_i$ has a $N(0, \sigma^2)$ where the variance $\sigma^2$ is unknown.
If $n = 15$, find the most powerful level $\alpha = 0.05$ test of $H_0 : \sigma^2 =$ 9 [redacted] versus $H_a : \sigma^2 =$ 25 [redacted]
Explicitly provide the Rejection Region (RR).

$$L(y_1, \ldots, y_{15} \mid \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} y_1^2} \times \cdots \times \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2} y_{15}^2}$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{15} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{15} y_i^2}$$

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{\left(\frac{1}{3\sqrt{2\pi}}\right)^{15} e^{-\frac{1}{6}\sum_{i=1}^{15} y_i^2}}{\left(\frac{1}{5\sqrt{2\pi}}\right)^{15} e^{-\frac{1}{10}\sum_{i=1}^{15} y_i^2}} \le k$$

$$\Rightarrow \frac{\cancel{\left(\frac{5\sqrt{2\pi}}{3\sqrt{2\pi}}\right)^{15}} e^{-1/6 \sum_{i=1}^{15} y_i^2 + 1/10 \sum_{i=1}^{15} y_i^2}}{\cancel{(5/3)^{15}}} \le \frac{k}{(5/3)^{15}} = k_1$$

$$\Rightarrow \ln\left(e^{\sum_{i=1}^{15} y_i^2 (-1/6 + 1/10)}\right) \le \ln(k_1) = k_2$$

$$\Rightarrow \frac{\sum_{i=1}^{15} y_i^2 (-1/6 + 1/10)}{(-1/6 + 1/10)} \le \frac{k_2}{(-1/6 + 1/10)} = k^*$$

change inequality b/c $(-1/6 + 1/10) < 0$.

$$\Rightarrow \sum_{i=1}^{15} y_i^2 \ge k^*$$

$$\alpha = P\left(\sum_{i=1}^{15} y_i^2 \ge k^* \mid \sigma^2 = 3\right) = P\left(\sum_{i=1}^{15} \frac{y_i^2}{\sigma^2} \ge \frac{k^*}{\sigma^2} \mid \sigma^2 = 3\right)$$

Recall: $\sum_{i=1}^{n} \left(\frac{y_i - \mu}{\sigma}\right)^2 \sim (N(0,1))^2 = \chi^2_{(n)}$

here, $\mu = 0$. Thus, $\sum_{i=1}^{15} \frac{y_i^2}{\sigma^2} \sim \chi^2_{(15)}$

and $\frac{k^*}{3} = \chi^2_{df=15,\ \alpha=0.05} = 24.9958$

Thus $RR = \{ \sum_{i=1}^{15} y_i^2 \ge \underbrace{24.9958(3)}_{74.9874} \}$ is the RR.