STA260 Tutorial 10 Question 3

Question 3

For some reason a lot of people like to assume measurement error is Normally distributed with a mean value μ and a standard deviation 4. Consider testing $H_0 : \mu = 0$ versus $H_a : \mu \neq 0$ based on $n = 16$ measurements. $\sigma^2 = 16$

Find the likelihood ratio test, λ, and its Rejection Region (RR) when $\alpha = 0.05$.

$$L(y_1, \dots y_{16} \mid \mu) = \left(\frac{1}{\sqrt{32\pi}}\right)^n e^{\frac{-1}{32}\sum_{i=1}^{16}(y_i-\mu)^2}$$

find the MLE: $\ell(\mu) = -\frac{n}{2}\ln(32\pi) - \frac{1}{32}\sum_{i=1}^{16}(y_i-\mu)^2$

$$\ell'(\mu) = -\frac{1}{32}(2)\sum_{i=1}^{16}(y_i-\mu)(-1) \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{1}{16}\sum_{i=1}^{16} y_i - \frac{1}{16}\sum_{i=1}^{16}\mu = 0$$

$$\Rightarrow \frac{1}{16}\sum_{i=1}^{16} y_i = \mu \Rightarrow \hat{\mu} = \bar{y}$$

verify it's a maximum:

$$\ell''(\mu) = 0 - \frac{1}{16}\sum_{i=1}^{16}(1) = -1 < 0$$

thus $\hat{\mu} = \bar{y}$ is the MLE.

Now, $$\lambda = \frac{L(\hat{\Omega}_0)}{L(\hat{\Omega})} = \frac{\cancel{\left(\frac{1}{\sqrt{32\pi}}\right)^{16}} e^{-\frac{1}{32}\sum_{i=1}^{16} y_i^2}}{\cancel{\left(\frac{1}{\sqrt{32\pi}}\right)^{16}} e^{-\frac{1}{32}\sum_{i=1}^{16}(y_i-\bar{y})^2}}$$

$$= e^{-\frac{1}{32}\sum_{i=1}^{16} y_i^2 + \frac{1}{32}\sum_{i=1}^{16}(y_i-\bar{y})^2}$$

RMK: $y_i^2 = ((y_i-\bar{y})+(\bar{y}-0))^2 = (y_i-\bar{y})^2 + 2(y_i-\bar{y})(\bar{y}) + \bar{y}^2$

$$\sum_{i=1}^{16} y_i^2 = \sum_{i=1}^{16}(y_i-\bar{y})^2 + 2\bar{y}\sum_{i=1}^{16}(y_i-\bar{y}) + \sum_{i=1}^{16}\bar{y}^2$$

$$= \sum_{i=1}^{16}(y_i-\bar{y})^2 + 2\bar{y}\Big(\underbrace{\sum_{i=1}^{16} y_i}_{16\bar{y}} - 16\bar{y}\Big) + 16\bar{y}^2$$

$$= \sum_{i=1}^{16}(y_i-\bar{y})^2 + 16\bar{y}^2$$

$$= e^{-\frac{1}{32}\left[\cancel{\sum_{i=1}^{16}(y_i-\bar{y})^2} + 16\bar{y}^2\right] + \cancel{\frac{1}{32}\sum_{i=1}^{16}(y_i-\bar{y})^2}} = e^{-\frac{1}{2}\bar{y}^2}$$

Hence, $\lambda = \ln\left(e^{-\frac{1}{2}\bar{y}^2}\right) \leq \ln(k) = k_1$

$-\frac{1}{2}\bar{y}^2 \leq k_1 \Rightarrow \bar{y}^2 \geq -2k_1 = k^*$

Recall: $\left(\frac{\bar{y}-\mu}{\sqrt{\sigma^2/n}}\right)^2 \sim \chi^2_{(1)} \Rightarrow \frac{n\bar{y}^2}{\sigma^2} \sim \chi^2_{(1)}$

Since $n=16$ and $\sigma^2=16$... we're good.

$\alpha = P(\bar{y}^2 \geq k^* \mid \mu = 0)$

$\chi^2_{df=16,\ \alpha=0.05} = 3.84146$

Thus $RR = \{\bar{y}^2 \geq 3.84146\}$