STA260 Tutorial 12 Question 1

Question 1

Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a random sample from a joint distribution $F_{X,Y}(x, y)$. Is $\hat{\sigma}_{X,Y} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)Y_i$ a consistent estimator of $Cov(X_i, Y_i)$?

$$\hat{\sigma}_{XY} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)Y_i = \frac{1}{n}\sum_{i=1}^{n} X_iY_i - \frac{1}{n}\bar{X}_n\sum_{i=1}^{n} Y_i$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_iY_i - \frac{1}{n}\bar{X}_n n\bar{Y}_n = 1/n\sum_{i=1}^{n} X_iY_i - \bar{X}_n\bar{Y}_n$$

By WLLN, $1/n\sum_{i=1}^{n} X_iY_i \xrightarrow{p} \mathbb{E}(X_iY_i)$

$$1/n\sum_{i=1}^{n} X_i \xrightarrow{p} \mathbb{E}(X_i)$$

$$1/n\sum_{i=1}^{n} Y_i \xrightarrow{p} \mathbb{E}(Y_i)$$

By cts mapping thm, $1/n\sum_{i=1}^{n} X_iY_i - \bar{X}_n\bar{Y}_n \xrightarrow{p} \mathbb{E}(X_iY_i) - \mathbb{E}(X_i)\mathbb{E}(Y_i)$

$$= Cov(X_i, Y_i)$$

Yes, it is a consistent estimator.