STA260 Tutorial 12 Question 2

Question 2

Let $X_1, \ldots, X_n$ be a random sample from a Normal(μ, σ^2) distribution.
Prove that $F = \frac{n(\bar{X}-\mu)^2}{S^2} \sim F(1, n-1)$

Note: $F(1,n-1)$ can be represented by $\dfrac{\chi^2_{(1)}/1}{\chi^2_{(n-1)}/n-1}$

Furtermore, $\dfrac{(\bar{x}-\mu)^2}{\sigma^2/n} \sim \chi^2_{(1)}$ and $\dfrac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Since $\bar{x}$ and s^2 are independent, clearly $\dfrac{(\bar{x}-\mu)^2}{\sigma^2/n} \perp\!\!\!\perp \dfrac{(n-1)s^2}{\sigma^2}$

Hence, $F(1,n-1) = \dfrac{\chi^2_{(1)}/1}{\chi^2_{(n-1)}/n-1} = \dfrac{\frac{(\bar{x}-\mu)^2}{\sigma^2/n}}{1} \div \dfrac{\frac{\cancel{(n-1)}s^2}{\sigma^2}}{\cancel{(n-1)}}$

$$= \frac{n(\bar{x}-\mu)^2}{\sigma^2} \div \frac{s^2}{\sigma^2} = \frac{n(\bar{x}-\mu)^2}{\cancel{\sigma^2}} \times \frac{\cancel{\sigma^2}}{s^2} = \frac{n(\bar{x}-\mu)^2}{s^2}$$

as desired.