STA260 Tutorial 12 Question 3

Question 3

Let $Y_i = x_i + E_i$ for $i = 1, ..., n$ where:

- $x_1, ..., x_n$ are fixed known constants.
- $E_1, ...E_n$ are i.i.d Normal$(0, \sigma^2)$ random variables, but σ^2 is unknown.
- Only $x_1, ..., x_n$ and $Y_1, ..., Y_n$ are observable.

(a) What is the distribution of Y_i?

(b) Find the MLE of σ^2. (Don't need to compute the second derivative test.)

(c) Is the estimator $\hat{\sigma}^2$ sufficient?

1. Note that $\mathbb{E}(Y_i) = \mathbb{E}(x_i + E_i) = x_i + 0 = x_i$

$\mathbb{V}(Y_i) = \mathbb{V}(x_i + E_i) = \mathbb{V}(E_i)$ since x_i is a constant.

$= \sigma^2$

Recall: if $Y_i \sim N(\mu_i, \sigma_i^2)$ then $\sum Y_i \sim N(\sum \mu_i, \sum \sigma_i^2)$

Hence since E_i's are normally distributed and x_i's are just constants, $Y_i \sim N(x_i, \sigma^2)$

2. $L(y_1, \dots, y_n | \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y_1 - x_1)^2} \times \dots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y_n - x_n)^2}$

$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{\frac{-1}{2\sigma^2}\sum(y_i - x_i)^2}$

$\ell(\sigma^2) = -n/2 \ln(2\pi) - n/2 \ln(\sigma^2) - \frac{1}{2\sigma^2}\sum(y_i - x_i)^2$

$\ell'(\sigma^2) = \frac{-n/2}{\sigma^2} + \frac{1}{2\sigma^4}\sum(y_i - x_i)^2 \overset{set}{=} 0$

$\Rightarrow n/2 = 1/2\sigma^2 \sum(y_i - x_i)^2 \Rightarrow \hat{\sigma}^2 = \frac{\sum(y_i - x_i)^2}{n}$ is the MLE.

3. Recall: $L(y_1, \dots, y_n | \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^n e^{\frac{-1}{2\sigma^2}\sum(y_i - x_i)^2}$

$= \left(\frac{1}{2\pi\sigma^2}\right)^n e^{\frac{-n}{2\sigma^2}\left(\sum\frac{(y_i - x_i)^2}{n}\right)}$

Let $h(y_1, \dots, y_n) = \left(\frac{1}{2\pi}\right)^{n/2}$

$g(U, \sigma^2) = \left(\frac{1}{\sigma^2}\right)^{n/2} e^{\frac{-n}{2\sigma^2}(U)}$, $U = \frac{\sum(y_i - x_i)^2}{n}$

YES, it is a sufficient estimator.