STA260 Tutorial 12 Question 5

Question 5

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from an exponential distribution with mean β where $0 < \beta < \infty$.

1. Find the MLE of β and use the second derivative test to prove it's a maximum.
2. Find the MLE of $P(Y \leq 10)$.

1\. $L(y_1, \dots, y_n | \beta) = \frac{1}{\beta} e^{-y_1/\beta} \times \cdots \times e^{-y_n/\beta} = \left(\frac{1}{\beta}\right)^n e^{-\frac{1}{\beta}\sum y_i}$

$\ell(\beta) = -n\ln(\beta) - \frac{1}{\beta}\sum y_i$

$\ell'(\beta) = -\frac{n}{\beta} + \frac{1}{\beta^2}\sum y_i \overset{\text{set}}{=} 0 \Rightarrow \beta n = \sum y_i$

$\Rightarrow \hat{\beta}_{MLE} = \frac{\sum y_i}{n} = \bar{y}$

$\ell''(\beta) = \frac{n}{\beta^2} - \frac{2}{\beta^3}\sum y_i \overset{\text{WTS}}{<} 0$

If $\frac{n}{\beta^2} - \frac{2}{\beta^3}\sum y_i < 0$ then $\frac{n}{\beta^2} < \frac{2}{\beta^3}\sum y_i$

$\beta n < 2\sum y_i \Rightarrow \beta < 2\bar{y}$ true b/c $\hat{\beta} = \bar{y} < 2\bar{y}$ for nonzero $\bar{y}$.

Thus it's a maximum.

2\. Recall: invariance property: if $\hat{\beta}_{MLE}$ is the MLE for β then $\forall$ function $g(x)$ where $g(x)$ is injective then the MLE of $g(\beta)$ is $g(\hat{\beta}_{MLE})$.

$P(Y \leq 10) = \int_0^{10} \frac{1}{\beta} e^{-y/\beta}\, dy = \frac{1}{\beta}\int_0^{10} e^{-y/\beta}\, dy = \frac{1}{\beta}\left[\frac{e^{-y/\beta}}{-1/\beta}\right]\Big|_0^{10}$

$= -\left[e^{-10/\beta} - 1\right] = 1 - e^{-10/\beta}$

note: e^x is an injective function. So same w/ $1 - e^{-10/x}$.

Hence, $1 - e^{-10/\bar{y}}$ is the MLE for $P(Y \leq 10)$ by the invariance property.