Question 2 (7.57 from the textbook) ← 7.58 for 7th Edition

Suppose that $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that the random variable:

$$U_n = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}}$$

satisfies the conditions of Theorem 7.4 and thus that the distribution of U_n converges to a standard normal distribution function as $n \to \infty$.

Hint: Consider $W_i = X_i - Y_i$ for $i = 1, 2, \ldots, n$.

Review Theorem 7.4 first!

By hint $W_i = X_i - Y_i$ for $i = 1, 2, 3 \ldots$

$\because E(X_i) = \mu_1$, $E(Y_i) = \mu_2$

$\therefore E(W_i) = E(X_i - Y_i) = E(X_i) - E(Y_i) = \mu_1 - \mu_2$

Also $\because Var(X_i) = \sigma_1^2$, $Var(Y_i) = \sigma_2^2$ And X_i and Y_i are independent.

$\therefore Var(W_i) = Var(X_i - Y_i) = Var(X_i) + Var(Y_i) = \sigma_1^2 + \sigma_2^2$

be careful about the sign.

For $\overline{W} = \dfrac{\sum_{i=1}^{n} W_i}{n} = \dfrac{\sum_{i=1}^{n}(X_i - Y_i)}{n} = \dfrac{\sum_{i=1}^{n} X_i}{n} - \dfrac{\sum_{i=1}^{n} Y_i}{n} = \overline{X} - \overline{Y}$

$\therefore E(\overline{W}) = E(\overline{X} - \overline{Y}) = \mu_1 - \mu_2$

and $Var(\overline{W}) = Var(\overline{X} - \overline{Y}) = Var(\overline{X}) + Var(\overline{Y})$

$Var(\overline{X}) = Var\left(\dfrac{\sum_{i=1}^{n} X_i}{n}\right) = \dfrac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) = \dfrac{1}{n^2} \cdot n \cdot \sigma_1^2 = \dfrac{\sigma_1^2}{n}$

Since X_i are independent

Similarly, $Var(\overline{Y}) = \dfrac{\sigma_2^2}{n}$

$\therefore Var(\overline{W}) = \dfrac{\sigma_1^2 + \sigma_2^2}{n}$

Since W_i are independent as well, $U_n = \dfrac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}} = \dfrac{\overline{W} - E(\overline{W})}{\sqrt{Var(\overline{W})}}$

this satisfies the condition of Theorem 7.4 and U_n converges to a standard normal distribution.