STA260 Tutorial 2 Question 3

Question 3

Let X and Y be two independent exponential random variables with mean 1. Show that $\frac{X}{Y}$ has an F distribution and find its degrees of freedom.

Hint: First prove $Exp(1) = Gamma(1, 1)$.

Note: $Gamma(\alpha=1, \beta=1)$

$$f(y) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} y^{\alpha-1} e^{-y/\beta} \qquad 0<y<\infty$$

$\Gamma(n) = (n-1)!$

$$= \frac{1}{\Gamma(1)(1)^{(1)}} y^{1-1} e^{-y/1}, \quad 0<y<\infty$$

$\Gamma(1) = (1-1)! = 0! = 1$

$$= e^{-y}, \quad 0<y<\infty$$

which is exponential.

Now, let $X \sim Gamma(1,1) \Rightarrow 2X \sim Gamma(1,2)$

note: if $X \sim Gamma(\alpha, \beta)$ $\qquad = Gamma(2/2, 2)$

then $c \in \mathbb{R}$, $cX \sim Gamma(\alpha, c\beta)$ $\qquad = \chi^2_{(2)}$

and $\chi^2_{(n)} = Gamma(n/2, 2)$

Hence, $2X \sim \chi^2_{(2)} \Rightarrow X \sim \chi^2_{(2)}/2$

similarly, $Y \sim \chi^2_{(2)}/2$

Thus since X & Y are indep:

$$\frac{X}{Y} = \frac{\chi^2_{(2)}/2}{\chi^2_{(2)}/2} \sim F(2,2)$$

(Recall: $F(n,m) \sim \dfrac{\chi^2_{(n)}/n}{\chi^2_{(m)}/m}$)