STA260 Tutorial 3 Question 2

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Question 2

Let $X_1, X_2, ..., X_n$ denote a random sample of size n from a distribution with the probability density function:

$$f(x) = \begin{cases} e^{-(x-\mu)} & x \geq \mu \\ 0 & \text{otherwise} \end{cases}$$

Let $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$ be an estimator for μ. Calculate $B(X_{(1)})$.

$$B(\hat{\mu}) = \mathbb{E}(\hat{\mu}) - \mu \qquad \mathbb{E}(\hat{\mu}) = \int_x^\infty x f_{\hat{\mu}}(x)\,dx$$

note: $F_{\hat{\mu}}(x) = P(\hat{\mu} \leq x)$

if $X_{(1)} > x$ then all $X_i > x$.

$$= P(\min\{X_1, X_2, \dots, X_n\} \leq x)$$

$$= 1 - P(\min\{X_1, X_2, \dots, X_n\} \geq x)$$

$$\overset{\text{ind}}{=} 1 - P(X_1 \geq x)P(X_2 \geq x) \dots P(X_n \geq x)$$

$$= 1 - [(1 - P(X_1 \leq x))(1 - P(X_2 \leq x)) \times \dots \times (1 - P(X_n \leq x))]$$

$$= 1 - [(1 - F_{X_1}(x))(1 - F_{X_2}(x)) \times \dots \times (1 - F_{X_n}(x))]$$

identically distributed

$$= 1 - (1 - F_{X_1}(x))^n$$

$$\frac{d F_{\hat{\mu}}(x)}{dx} = -n(1 - F_{X_1}(x))^{n-1}(-f_{X_1}(x))$$

$$= n(1 - F_{X_1}(x))^{n-1} f_{X_1}(x) = f_{\hat{\mu}}(x)$$

$$F_{X_1}(x) = \int_\mu^x e^{-(t-\mu)}\,dt = e^\mu \int_\mu^x e^{-t}\,dt = e^\mu\left(-e^{-t}\Big|_\mu^x\right)$$

$$= e^\mu(-e^{-x} + e^{-\mu}) = 1 - e^{-x+\mu}$$

Hence, $f_{\hat{\mu}}(x) = n(1 - (1 - e^{-x+\mu}))^{n-1}(e^{-x+\mu})$

$$= n(e^{-x+\mu})^{n-1}(e^{-x+\mu})$$

$$= n(e^{-xn})(e^{\mu n})$$

$$\mathbb{E}(\hat{\mu}) = \int_x^\infty x f_{\hat{\mu}}(x)\,dx = \int_\mu^\infty x \cdot n e^{-xn} e^{\mu n}\,dx$$

$$= n e^{n\mu} \int_\mu^\infty x e^{-xn}\,dx$$

integration by parts: $uv - \int v\,du$

LIATE

$u = x$ $\qquad dv = e^{-xn}dx$

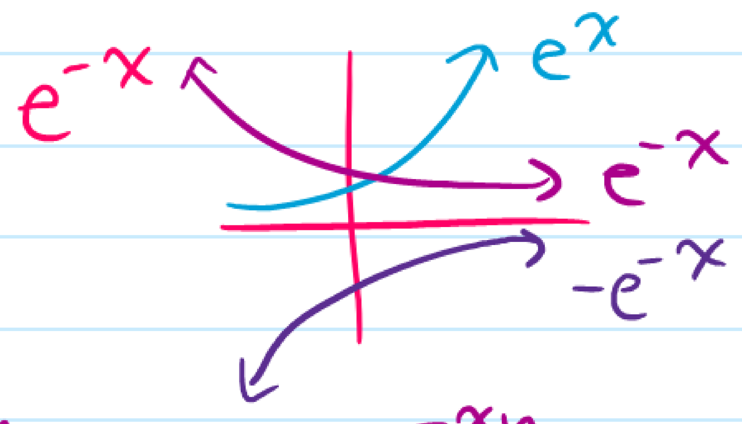
$du = dx$ $\qquad v = -\frac{e^{-xn}}{n}$

$$= ne^{n\mu}\left[\frac{-xe^{-xn}}{n} + \int \frac{e^{-xn}}{n}dx\right]\Bigg|_\mu^\infty$$

'' $-\frac{e^{-xn}}{n^2}$

$$= ne^{n\mu}\left[\lim_{x\to\infty} \frac{-xe^{-xn}}{n} - \frac{e^{-xn}}{n^2} + \frac{\mu e^{-\mu n}}{n} + \frac{e^{-\mu n}}{n^2}\right]$$

note:



hence $\lim_{x\to\infty} -\frac{e^{-xn}}{n} = 0$

and $\lim_{x\to\infty} \frac{-xe^{-xn}}{n} = \frac{1}{n}\lim_{x\to\infty} \frac{-x}{e^{xn}} = \frac{1}{n}(0) = 0$

$$= n\cancel{e^{n\mu}}\left[\frac{\mu \cancel{e^{-n\mu}}}{n} + \frac{\cancel{e^{-n\mu}}}{n^2}\right] = \mu + 1/n \quad \text{(YAY!)}$$

THUS: $\beta(\hat{\mu}) = \mathbb{E}(\hat{\mu}) - \mu = \mu + 1/n - \mu = 1/n$