STA260 Tutorial 3 Question 3

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Question 3

Let $Y_1, Y_2, ..., Y_n$ be a random sample from a normal distribution with mean μ and variance $\sigma^2 > 0$ and let us define the sample variance as follows:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

Prove that the estimator

$$\hat{\sigma} = \sqrt{\frac{n-1}{2}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}S \quad \star$$

is unbiased for estimating σ, where S is the square root of the sample variance.

Hint: prove that if $Y \sim Gamma(\alpha, \beta)$ then if $\alpha + a > 0$, then

$$E[Y^a] = \frac{\beta^a\Gamma(\alpha + a)}{\Gamma(\alpha)}$$

Then recall the relationship between the χ^2 and $Gamma$ distributions.

Pf of hint:

$$\mathbb{E}(Y^a) = \int_0^\infty y^a \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}\,dy$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{a+\alpha-1} e^{-y/\beta}\,dy$$

multiply by 1: $\frac{\Gamma(a+\alpha)\beta^{a+\alpha}}{\Gamma(a+\alpha)\beta^{a+\alpha}}$

$$= \frac{\Gamma(a+\alpha)\beta^{a+\alpha}}{\Gamma(a+\alpha)\beta^{a+\alpha}} \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{a+\alpha-1} e^{-y/\beta}\,dy$$

want: SWAP

$$= \frac{\Gamma(a+\alpha)\beta^{a+\cancel{\alpha}}}{\Gamma(\alpha)\cancel{\beta^\alpha}} \boxed{\int_0^\infty \frac{1}{\Gamma(a+\alpha)\beta^{a+\alpha}} y^{a+\alpha-1} e^{-y/\beta}\,dy}$$

pdf of Gamma$(a+\alpha, \beta)$

Recall: $\int_{-\infty}^\infty f_Y(y)\,dy = 1$ if $f_Y(y)$ is a valid pdf.

$$= \frac{\Gamma(a+\alpha)\beta^a}{\Gamma(\alpha)} \text{ as desired.}$$

Back to Q...

Recall: $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)} = \text{Gamma}(\alpha = \frac{n-1}{2}, \beta = 2)$

because of $(\star)$, try $a = 1/2$.

Then, $$\mathbb{E}\left(\left(\frac{(n-1)S^2}{\sigma^2}\right)^{1/2}\right) = \frac{2^{(1/2)}\Gamma(\underset{\text{“}\alpha\text{”}}{\frac{n-1}{2}} + \underset{\text{“}a\text{”}}{\frac{1}{2}})}{\Gamma(\frac{n-1}{2})}$$

$$\mathbb{E}\left(\frac{\sqrt{n-1}\,S}{\sigma}\right) = \frac{2^{1/2}\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}$$

$$\frac{1}{\sigma}\mathbb{E}(\sqrt{n-1}\,S) = \boxed{\frac{2^{1/2}\Gamma(n/2)}{\Gamma(\frac{n-1}{2})}}$$

$$\mathbb{E}\left(\sqrt{\frac{n-1}{2}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(n/2)}S\right) = \sigma \quad \text{as desired.}$$