STA260 Tutorial 3 Question 4

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Question 4 (8.20 from the textbook)

Suppose that Y_1, Y_2, Y_3, Y_4 denote a random sample of size 4 from a population with an exponential distribution whose density is given by:

$$f(y) = \begin{cases} \frac{1}{\theta} e^{\frac{-y}{\theta}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $X = \sqrt{Y_1 Y_2}$. Find a multiple of X that is an unbiased estimator for θ.

Let's experiment: $\mathbb{E}(X) = \mathbb{E}(\sqrt{Y_1 Y_2}) \overset{\text{ind}}{=} \mathbb{E}(\sqrt{Y_1})\mathbb{E}(\sqrt{Y_2})$

now, $\exp(\beta) = \text{Gamma}(\alpha = 1, \beta)$. Thus,

$Y_1 \sim \exp(\theta) = \text{Gamma}(\alpha = 1, \theta)$

Recall: $\mathbb{E}(Y^a) = \dfrac{\beta^a \Gamma(a+\alpha)}{\Gamma(\alpha)}$ for $Y \sim \text{Gamma}(\alpha, \beta)$

Recall: $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$

$\Gamma(1/2 + 1) = 1/2\, \Gamma(1/2)$

and $\Gamma(n) = (n-1)!\ , n \in \mathbb{N}$

$\Gamma(1) = (1-1)! = 0! = 1$

$\Gamma(1/2) = \sqrt{\pi}$

Thus, $\mathbb{E}(Y_1^{1/2}) = \dfrac{\theta^{1/2}\Gamma(1/2 + 1)}{\Gamma(1)}$

$= \dfrac{\theta^{1/2}(1/2)\Gamma(1/2)}{1} = \theta^{1/2}\dfrac{\sqrt{\pi}}{2}$

Hence $\mathbb{E}(\sqrt{Y_1})\mathbb{E}(\sqrt{Y_2})$

$= \theta^{1/2}\left(\dfrac{\sqrt{\pi}}{2}\right)\theta^{1/2}\left(\dfrac{\sqrt{\pi}}{2}\right)$

$= \theta\left(\dfrac{\pi}{4}\right)$

Hence $\mathbb{E}(X) = \theta\dfrac{\pi}{4}$. Want: $\mathbb{E}(cX) = \theta$

solve for c: $c\,\mathbb{E}(X) \overset{\text{set}}{=} \theta$

$c\left(\cancel{\theta}\dfrac{\pi}{4}\right) = \cancel{\theta}$

$\Rightarrow \quad c = 4/\pi$

Hence $4/\pi\, X$ produces an unbiased est. for θ.