Question 1 (8.44 from the textbook)

Let Y have probability density function

$$f_Y(t) = \begin{cases} \frac{2(\theta - t)}{\theta^2} & 0 < t < \theta \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Show that Y has a distribution function

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{2y}{\theta} - \frac{y^2}{\theta^2} & 0 < y < \theta \\ 1 & y \geq \theta \end{cases}$$

(b) Show that $\frac{Y}{\theta}$ is a pivotal quantity.

(c) Use the pivotal quantity from part (b) to find a 90% lower confidence limit for θ.

(a) we consider the interval $0 < t < \theta$ since $f_{Y}(t) = 0$ otherwise.

$$F_Y(y) = P(Y \leq y) = \int_0^y \frac{2(\theta - t)}{\theta^2}\,dt$$

$$= \frac{1}{\theta^2}\int_0^y 2\theta - 2t\,dt$$

$$= \frac{1}{\theta^2}\left(2\theta t - t^2\right)\Big|_0^y$$

$$= \frac{1}{\theta^2}\left(2\theta y - y^2\right) = \frac{2y}{\theta} - \frac{y^2}{\theta^2} \quad \text{for } 0 < y < \theta$$

Therefore, $F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \frac{2y}{\theta} - \frac{y^2}{\theta^2} & 0 < y < \theta \\ 1 & y \geq \theta \end{cases}$

cumulated all possible probabilities.

(b) Let $W = \frac{Y}{\theta}$

$$F_W(w) = P(W \le w) = P\left(\frac{Y}{\theta} \le w\right) = P(Y \le \theta w)$$

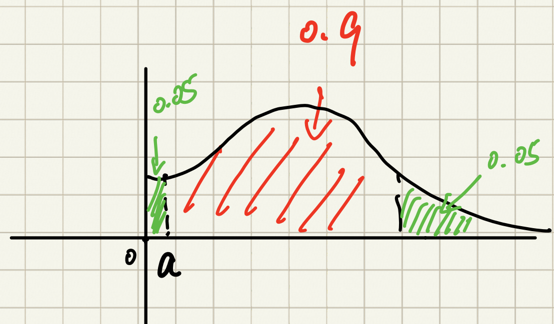
Since $P(Y \le \theta w) = F_Y(\theta w) = \frac{2 \cdot \theta w}{\theta} - \frac{(\theta w)^2}{\theta^2}$

$= 2w - w^2$ for $0 < w < 1$.

since $0 < \theta w < \theta$

W doesn't depend on θ, therefore $\frac{Y}{\theta}$ is a pivotal quantity.

(c) Assume $P(W \le a) = F_W(a) = 2a - a^2 = 0.05$



By quadratic formula. we can get

$$a = \frac{2 \pm \sqrt{4 - 4 \times 0.05}}{2} \Rightarrow a_1 \approx 1.97468$$

$a_2 \approx 0.0253206$ choosing a_2 since $0 < w < 1$ from part b.

$\because W = \frac{Y}{\theta}$

Therefore, the 90% lower confident limit for θ is $\frac{Y}{0.0253206}$