**Question 2** (8.125 from the textbook)

Suppose that independent samples of sizes $n_1$ and $n_2$ are taken from two normally distributed populations with variances $\sigma_1^2$ and $\sigma_2^2$, respectively. If $S_1^2$ and $S_2^2$ denote the respective sample variances, Theorem 7.3 implies that $(n_1-1)S_1^2/\sigma_1^2$ and $(n_2-1)S_2^2/\sigma_2^2$ have $\chi^2$ distributions with $n_1-1$ and $n_2-1$ degrees of freedom, respectively. Further, these $\chi^2$ distributed random variables are independent because the samples were independently taken.

1. Use these quantities to construct a random variable that has an $F$ distribution with $n_1-1$ numerator degrees of freedom and $n_2-1$ denominator degrees of freedom.
2. Use the $F$-distributed quantity from part (a) as a pivotal quantity and derive a formula for a $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$.

Review Definition 7.3 !

1. By Definition 7.3 about F distribution.

$$F = \frac{\dfrac{(n_1-1)S_1^2}{\sigma_1^2} \Big/ n_1-1}{\dfrac{(n_2-1)S_2^2}{\sigma_2^2} \Big/ n_2-1}$$

$$= \frac{\dfrac{S_1^2}{\sigma_1^2}}{\dfrac{S_2^2}{\sigma_2^2}} = \frac{S_1^2 \cdot \sigma_2^2}{S_2^2 \cdot \sigma_1^2}$$

2. $P(F_{1-\frac{\alpha}{2}} < F < F_{\frac{\alpha}{2}}) = 1-\alpha$

$$\Rightarrow P\left(F_{1-\frac{\alpha}{2}} < \frac{S_1^2 \cdot \sigma_2^2}{S_2^2 \cdot \sigma_1^2} < F_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$\Rightarrow P\left(\frac{S_2^2}{S_1^2} \cdot F_{1-\frac{\alpha}{2}} < \frac{\sigma_2^2}{\sigma_1^2} < \frac{S_2^2}{S_1^2} \cdot F_{\frac{\alpha}{2}}\right) = 1-\alpha$$

Therefore the $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$ is $\left(\frac{S_2^2}{S_1^2} F_{1-\frac{\alpha}{2}}, \frac{S_2^2}{S_1^2} F_{\frac{\alpha}{2}}\right)$.