Question 3 (8.129 from the textbook)

If

$$S_\star^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n} \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1}$$

then $S_\star^2$ is a biased estimator of σ^2, but S^2 is an unbiased estimator for the same parameter. If we sample from a normal population,

(a) Find $\mathbb{V}(S_\star^2)$

(b) Prove $\mathbb{V}(S^2) > \mathbb{V}(S_\star^2)$

As we know $S^2 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})}{n-1} \Rightarrow \frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})}{\sigma^2} \sim \chi^2_{n-1}$

$$\therefore Var\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{(n-1)^2}{(\sigma^2)^2} Var(S^2) = 2(n-1)$$

$$\Rightarrow Var(S^2) = \frac{2(n-1)\cdot\sigma^4}{(n-1)^2} = \frac{2\sigma^4}{n-1}$$

(a) $\because S_*^2 = \frac{S^2(n-1)}{n}$

$$\therefore Var(S_*^2) = Var\left(\frac{S^2(n-1)}{n}\right) = \frac{(n-1)^2}{n^2} Var(S^2) = \frac{(n-1)^2}{n^2}\cdot\frac{2\sigma^4}{n-1}$$

$$= \frac{2(n-1)\sigma^4}{n^2}$$

(b) $Var(S_*^2) = Var\left(\frac{n-1}{n}\cdot S^2\right)$

$$\Rightarrow Var(S_*^2) = \frac{(n-1)^2}{n^2} Var(S^2)$$

Since $0 < \frac{(n-1)^2}{n^2} < 1$

Therefore, $Var(S_*^2) < Var(S^2)$