Question 1 (9.5 ~~from~~ from the textbook) ← 9.3 for 7th Edition.

If $Y_1, Y_2, ..., Y_n$ denote a random sample from the $Uniform(\theta, \theta+1)$ distribution, then both

$$\hat{\theta}_1 = \bar{Y} - \frac{1}{2} \quad \text{and} \quad \hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}$$

are unbiased estimators for θ. Which one is the better unbiased estimator? Calculate the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

Review how to calculate efficiency!

$E(\hat{\theta}_1) = E(\bar{Y} - \frac{1}{2}) = E(\bar{Y}) - \frac{1}{2} = \frac{\theta+\theta+1}{2} - \frac{1}{2} = \theta$, $\hat{\theta}_1$ unbiased estimator

Since independent

$Var(\hat{\theta}_1) = Var(\bar{Y} - \frac{1}{2}) = Var(\bar{Y}) = \frac{1}{n^2}\sum_{i=1}^{n} Var(Y_i) = \frac{1}{n}\frac{(\theta+1-\theta)^2}{12} = \frac{1}{12n}$

And $Y_{(n)}$ are ordered random variable.

So we can get $F_{Y_{(n)}}(y) = P(Y_{(n)} \le y)$

$= P(Y_1 \le y)P(Y_2 \le y) \ldots P(Y_n \le y)$

$= [F(y)]^n$

And pdf $g_n(y) = n[F(y)]^{n-1} \cdot F'(y) = n \cdot [F(y)]^{n-1} \cdot f(y)$

So pdf for $Y_{(n)}$ is $n(y-\theta)^{n-1}$ for $\theta < y < \theta+1$

Note. pdf for Uniform Distribution is $\frac{1}{\theta_2 - \theta_1}$, for this question the pdf is $\frac{1}{\theta+1-\theta} = 1$.

CDF is $\int_\theta^y 1\,dw = w\Big|_\theta^y = y - \theta$.

$E(\hat{\theta}_2) = E(Y_{(n)} - \frac{n}{n+1}) = E(Y_{(n)}) - \frac{n}{n+1}$

$E(Y_{(n)}) = \int_\theta^{\theta+1} n(y-\theta)^{n-1} \cdot y\,dy = n\int_\theta^{\theta+1} y(y-\theta)^{n-1}\,dy$

Hint. Integration by parts.

Using the Integration by parts $u \cdot dv = u \cdot v - v \cdot du$

Let $u = y$ $\qquad$ $v = \frac{(y-\theta)^n}{n}$

$du = dy$ $\qquad$ $dv = (y-\theta)^{n-1} dy$

$$y \cdot \frac{(y-\theta)^n}{n} \Big|_{\theta}^{\theta+1} - \int_{\theta}^{\theta+1} \frac{(y-\theta)^n}{n} dy$$

$$= y \cdot \frac{(y-\theta)^n}{n} - \frac{(y-\theta)^{n+1}}{n \cdot (n+1)} \Big|_{\theta}^{\theta+1}$$

$$= \frac{(\theta+1)}{n} - \frac{1}{n(n+1)}$$

Then $E(Y_{(n)}) = n\left(\frac{\theta+1}{n} - \frac{1}{n(n+1)}\right) = \theta+1 - \frac{1}{n+1} = \theta + \frac{n}{n+1}$

Then $E(\hat{\theta}_2) = E(Y_{(n)}) - \frac{n}{n+1} = \theta.$, $\hat{\theta}_2$ unbiased estimator.

$$Var(\hat{\theta}_2) = Var(Y_{(n)} - \frac{n}{n+1}) = Var(Y_{(n)}) = E(Y_{(n)}^2) - (E(Y_{(n)}))^2$$

$E(Y_{(n)}^2) = \int_{\theta}^{\theta+1} n \cdot y^2 (y-\theta)^{n-1} dy$ $\Leftarrow$ Similarly, Using Integration by parts. Twice !!

$$= n\left(\frac{y^2 (y-\theta)^n}{n} \Big|_{\theta}^{\theta+1} - \int_{\theta}^{\theta+1} 2 \frac{(y-\theta)^n}{n} y \, dy\right)$$

$$= (\theta+1)^2 - 2\int_{\theta}^{\theta+1} (y-\theta)^n y \, dy$$

$$= (\theta+1)^2 - 2\left(\frac{\theta+1}{n+1} - \frac{1}{(n+1)(n+2)}\right)$$

Then $Var(\hat{\theta}_2) = (\theta+1)^2 - \dfrac{2(\theta+1)}{n+1} + \dfrac{2}{(n+1)(n+2)} - \left(\theta + \dfrac{n}{n+1}\right)^2$

$=$ Doing some algebra...

$$= \frac{n}{(n+2)(n+1)^2}$$

Finally, the $eff(\hat{\theta}_2, \hat{\theta}_1) = \dfrac{Var(\hat{\theta}_2)}{Var(\hat{\theta}_1)}$

$$= \frac{n}{(n+2)(n+1)^2} \cdot \frac{12n}{1}$$

$$= \frac{12n^2}{(n+2)(n+1)^2} \quad \blacksquare$$