Question 2 (9.30 from the textbook)

Let $Y_1, Y_2, ..., Y_n$ be independent random variables, each with the probability density function:

$$f(y) = \begin{cases} 3y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\bar{Y}$ converges in probability to some constant and state which exact constant.

We can see $f(y)$ follows Beta Distribution

$$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1} \text{ when } \alpha = 3, \beta = 1$$

$$\text{mean} = \frac{\alpha}{\alpha+\beta} = \frac{3}{4}$$

$$\text{Variance} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{3}{16 \cdot 5} = \frac{3}{80}$$

$$E(\bar{Y}) = E\left(\frac{\sum_{i=1}^{n} Y_i}{n}\right) = \frac{1}{n} \cdot \sum_{i=1}^{n} E(Y_i) = \frac{1}{n} \cdot n \cdot E(Y_i) = \frac{3}{4}$$

$$Var(\bar{Y}) = Var\left(\frac{\sum_{i=1}^{n} Y_i}{n}\right) = \frac{1}{n^2} \cdot \sum_{i=1}^{n} Var(Y_i) = \frac{Var(Y_i)}{n} = \frac{3}{80n}$$

Since Y_i are independent

$$\lim_{n \to \infty} \frac{3}{80n} = 0$$

Therefore, $\bar{Y}$ is a consistent estimator and $\bar{Y}$ converges in probability to $\frac{3}{4}$.