Question 3 (9.34 from the textbook)

Let Rayleigh density function is given by:

$$f(y) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{\frac{-y^2}{\theta}} & y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

From a previous exercise (6.34) it is proven that Y^2 has an exponential distribution with mean θ. If $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Rayleigh distribution, prove that $W_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is a consistent estimator for θ.

Since Y^2 has an exponential distribution with mean θ,

$E(Y^2) = \theta$, $Var(Y^2) = \theta^2$

Then $E(W_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right)$

$$= \frac{1}{n}\sum_{i=1}^{n} E(Y_i^2) = \frac{1}{n} \cdot n \cdot \theta = \theta$$

$$Var(W_n) = Var\left(\frac{1}{n}\sum_{i=1}^{n} Y_i^2\right) = \frac{1}{n^2} \cdot Var\left(\sum_{i=1}^{n} Y_i^2\right)$$

$$= \frac{1}{n^2} \cdot \sum_{i=1}^{n} Var(Y_i^2)$$

$$= \frac{1}{n^2} \cdot n \cdot \theta^2 = \frac{\theta^2}{n}$$

Since independent.

By Theorem 9.1, $\lim_{n\to\infty} Var(W_n) = \lim_{n\to\infty} \frac{\theta^2}{n} = 0$

Therefore, $W_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is a consistent estimator for θ.