Question 3

Let $Y_1, ..., Y_n$ be a random sample from a $Bernoulli(p)$ distribution. Find the MVUE of $(1-p)^2$.

ind

Recall: for a Bernoulli dist:

$P(Y_i=1)=p \quad P(Y_i=0)=\underbrace{(1-p)}_{\text{want: squared version of this}} \qquad \mathbb{E}(Y_i)=p$

Since the distribution is known, we should

1. find an unbiased estimator U_1
2. find a sufficient estimator U_2

such that $\mathbb{E}(U_1 | U_2)$ is the MVUE.

Consider: $U_1 = \begin{cases} 1 & Y_1=0, Y_2=0 \\ 0 & Y_1\neq 0 \text{ OR } Y_2\neq 0 \end{cases}$

Then $\mathbb{E}(U_1) = 1 \times P(Y_1=0, Y_2=0) + 0 \times P(Y_1\neq 0 \vee Y_2 \neq 0)$

$= P(Y_1=0, Y_2=0)$

$\stackrel{ind}{=} P(Y_1=0)P(Y_2=0)$

$=(1-p)^2$

So U_1 is an unbiased est. for $(1-p)^2$.

Find sufficient statistic:

$L(y_1,\dots,y_n|p) = p^{y_1}(1-p)^{1-y_1} \times \dots \times p^{y_n}(1-p)^{1-y_n}$

$= p^{\sum y_i}(1-p)^{\sum 1-y_i}$

$= p^{\sum y_i}(1-p)^{n-\sum y_i}$ → try: $U_2 = \sum y_i$

$= (1-p)^n \left(\frac{p}{1-p}\right)^{\sum y_i}$

let $h(y_1,\dots y_n)=1$, $g(u,p)=(1-p)^n\left(\frac{p}{1-p}\right)^{\sum y_i}$ then $\sum y_i$ is sufficient.

RECALL: $\text{Bernoulli}(p) = \text{Binomial}(n=1,p)$

If $Y_i \stackrel{iid}{\sim} \text{Bernoulli}(p)$ then $\sum_{i=1}^{n} Y_i \sim \text{Binomial}(n,p)$

Now, $\mathbb{E}(U_1|U_2) = 1 \times P(Y_1=0, Y_2=0 | U_2=\sum Y_i) + 0 \times P(Y_1\neq 0 \vee Y_2\neq 0 | U_2 = \sum Y_i)$

$= P(Y_1=0, Y_2=0 | U_2=\sum Y_i)$

$= \dfrac{P(Y_1=0, Y_2=0, U_2=\sum Y_i)}{P(U_2=\sum Y_i)}$ → tells you it has a binom distribution.

$= \dfrac{(1-p)^2 \binom{n-2}{u} p^u (1-p)^{(n-2)-u}}{\binom{n}{u} p^u (1-p)^{n-u}}$

Okay final answer!

The simplification doesn't necessarily look good...

$= \boxed{\binom{n-2}{u} \Big/ \binom{n}{u}} = \dfrac{(n-2)!}{\cancel{u!}(n-2-u)!} \times \dfrac{\cancel{u!}(n-u)!}{n!}$

$= \dfrac{\cancel{(n-2)!}}{\cancel{(n-2-u)!}} \times \dfrac{(n-u)(n-u-1)\cancel{(n-u-2)!}}{n(n-1)\cancel{(n-2)!}}$

$= \dfrac{(n-u)(n-u-1)}{n(n-1)}$ is the MVUE.