Question 4

Let $Y_1, \ldots, Y_n$ be a random sample from a $Gamma(2, \beta)$ distribution. Find the UMVUE of $\beta(\beta+2)$. $= \beta^2 + 2\beta$

1. Let's find the unbiased estimator of $\beta^2 + 2\beta$.

Rmk: $\mathbb{E}(Y) = \alpha\beta = 2\beta \quad V(Y) = \alpha\beta^2 = 2\beta^2$

$\mathbb{E}(\bar{Y}) = \mathbb{E}(\frac{1}{n}\sum Y_i) = \frac{1}{n}\sum \mathbb{E}(Y_i) = \frac{1}{n}\sum 2\beta = 2\beta$

$V(\bar{Y}) = V(\frac{1}{n}\sum Y_i) \overset{ind}{=} \frac{1}{n^2}\sum V(Y_i) = \frac{1}{n^2}\sum(2\beta^2) = \frac{2\beta^2}{n}$

Try: $\mathbb{E}(\bar{Y}^2) = Var(\bar{Y}) + (\mathbb{E}(\bar{Y}))^2$ want: β^2

$$= \frac{2\beta^2}{n} + (2\beta)^2 = \beta^2\left(\frac{2}{n} + 4\right) = \beta^2\left(\frac{2+4n}{n}\right)$$

Then $\mathbb{E}\left(\left(\frac{n}{2+4n}\right)\bar{Y}^2\right) = \beta^2$

Thus $\mathbb{E}\left(\left(\frac{n}{2+4n}\right)\bar{Y}^2 + \bar{Y}\right) = \mathbb{E}\left[\left(\frac{n}{2+4n}\right)\bar{Y}^2\right] + \mathbb{E}(\bar{Y})$

$$= \beta^2 + 2\beta$$

so $\left(\frac{n}{2+4n}\right)\bar{Y}^2 + \bar{Y}$ is our unbiased estimator.

2. Let's find the complete & sufficient statistic.

$$f(y|\beta) = \frac{1}{\Gamma(2)\beta^2} y e^{-y/\beta} = y\beta^{-2}e^{-y/\beta} = e^{-y/\beta + \ln(y) + \ln(\beta^{-2})}$$

$\Gamma(2) = (2-1)! = 1! = 1$

compare to $f(y|\theta) = e^{p(\theta)u(y) + q(\theta) + s(y)}$

where $U = \sum_{i=1}^{n} u(y_i)$ is sufficient & complete.

Here, $p(\theta) = -1/\beta \quad u(y) = y \quad q(\theta) = \ln(\beta^{-2}) \quad s(y) = \ln(y)$

Thus $U = \sum_{i=1}^{n} Y_i$ is sufficient & complete.

Finally, $\mathbb{E}\left(\left(\frac{n}{2+4n}\right)\bar{Y}^2 + \bar{Y} \,\middle|\, \sum Y_i\right)$

$$= \mathbb{E}\left(\left(\frac{n}{2+4n}\right)\left(\frac{1}{n}\sum Y_i\right)^2 + \left(\frac{1}{n}\sum Y_i\right) \,\middle|\, \sum Y_i\right)$$

$= \left(\frac{n}{2+4n}\right)\left(\frac{1}{n}\sum Y_i\right)^2 + \left(\frac{1}{n}\sum Y_i\right)$ is the UMVUE.