STA260 Tutorial 8 Question 1

Question 1

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with common probability mass function

$$p_Y(y \mid \theta) = \theta^y (1-\theta)^{1-y}, \quad y = 0 \text{ or } 1,$$

where $0 \le \theta \le 1$ is a parameter.

(a) Derive the Fisher information $I_n(\theta)$ of this distribution.

(b) Prove that $\bar{Y}$ is MVUE of θ using the Cramer-Rao theorem.

a) Recall: $I_n(\theta) = n I(\theta) = n\left(-E\left(\frac{\partial^2}{\partial^2\theta} \ln(P_Y(y|\theta))\right)\right)$

$\ln(P_Y(y|\theta)) = \ln(\theta^y (1-\theta)^{1-y}) = y\ln(\theta) + (1-y)\ln(1-\theta)$

$\frac{\partial \ln(P_Y(y|\theta))}{\partial\theta} = y\left(\frac{1}{\theta}\right) + \frac{(1-y)(-1)}{1-\theta} = y/\theta - \left(\frac{1-y}{1-\theta}\right)$

$\frac{\partial^2 \ln(P_Y(y|\theta))}{\partial^2\theta} = \frac{-y}{\theta^2} - \frac{(1-y)}{(1-\theta)^2}$

$E\left(\frac{\partial^2 \ln(P_Y(y|\theta))}{\partial^2\theta}\right) = \frac{-1}{\theta^2} E(y) - \frac{1}{(1-\theta)^2} E(1-y)$

$E(y) = 0 \cdot \theta^0 (1-\theta)^{1-0} + 1 \cdot \theta^1 (1-\theta)^{1-1} = \theta$

$E(1-y) \overset{\text{lin}}{=} 1 - E(y) = 1-\theta$

$= \frac{-1}{\theta^2}(\theta) - \frac{1}{(1-\theta)^2}(1-\theta) = \frac{-1}{\theta} - \frac{1}{1-\theta}$

$I(\theta) = 1/\theta + 1/(1-\theta) = \frac{1}{\theta(1-\theta)}$

Thus $I_n(\theta) = \frac{n}{\theta(1-\theta)}$.

b) 1. WTShow $\bar{Y}$ is unbiased for θ.

$E(\bar{Y}) = \frac{1}{n}\sum E(Y_i) = \frac{1}{n} n\theta = \theta$

2. WTShow $Var(\bar{Y}) = \frac{1}{I_n(\theta)}$

$E(y^2) = \sum y^2 P_Y(y) = 0^2\theta^0(1-\theta)^{1-0} + 1^2\theta^1(1-\theta)^{1-1} = \theta$

$Var(Y) = E(y^2) - E(y)^2 = \theta - \theta^2 = \theta(1-\theta)$

$Var(\bar{Y}) \overset{\text{ind}}{=} \frac{1}{n^2}\sum V(Y) = \frac{1}{n^2}\sum \theta(1-\theta) = \frac{\theta(1-\theta)}{n}$

$= 1/I_n(\theta)$

Hence by CR thm, θ is the MVUE for θ.