STA260 Tutorial 8 Question 2

Question 2

Let $Y_1, ..., Y_n$ denote a random sample from $N(0, \theta^2)$ where $\theta > 0$ is unknown. Compute the Cramer-Rao Lower Bound.

$$f(y_i|\theta) = \frac{1}{\theta\sqrt{2\pi}} e^{\frac{-1}{2\theta^2}(y_i)^2}$$

$$\ln\left(f(y_i|\theta)\right) = -\ln(\theta\sqrt{2\pi}) - \frac{1}{2\theta^2}(y_i)^2$$

$$\frac{\partial \ln f(y_i|\theta)}{\partial\theta} = \frac{-\sqrt{2\pi}}{\theta\sqrt{2\pi}} + \frac{2}{2\theta^3}(y_i)^2$$

$$= -1/\theta + (y_i)^2/\theta^3$$

$$\frac{\partial^2 \ln f(y_i|\theta)}{\partial^2\theta} = 1/\theta^2 - \frac{3(y_i)^2}{\theta^4}$$

note:

$$E(y_i^2) = V(y_i) + E(y_i)^2$$

$$= \theta^2 + 0^2 = \theta^2$$

$$E\left(\frac{\partial^2 \ln f(y_i|\theta)}{\partial^2\theta}\right) = \frac{1}{\theta^2} - \frac{3E(y_i^2)}{\theta^4}$$

$$= 1/\theta^2 - \frac{3\theta^2}{\theta^4} = 1/\theta^2 - 3/\theta^2 = -2/\theta^2$$

$$I(\theta) = -E\left(\frac{\partial^2}{\partial^2\theta}\ln f(y_i|\theta)\right) = 2/\theta^2$$

$$I_n(\theta) = nI(\theta) = 2n/\theta^2$$

$$\text{CR-lower bound} = 1/I_n(\theta) = \frac{\theta^2}{2n}$$