STA260 Tutorial 9 Question 1

Question 1

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from a distribution with the following probability density function with parameters $\alpha > 0$ and $\beta > 0$, where β is known.

$$f(y|\alpha,\beta) = \begin{cases} \alpha\beta^{\alpha}y^{-(\alpha+1)} & y \geq \beta \\ 0 & \text{otherwise} \end{cases}$$

Find the MLE for α.

note: Support depends on β... But NOT on α!

Hence we can solve this using calculus.

$$L(y_1, \dots y_n | \alpha) = \alpha\beta^{\alpha} y_1^{-(\alpha+1)} \times \dots \times \alpha\beta^{\alpha} y_n^{-(\alpha+1)}$$

$$= \alpha^n \beta^{\alpha n} \left(\prod_{i=1}^{n} y_i\right)^{-(\alpha+1)}$$

$$\ell(\alpha) = n\ln(\alpha) + \alpha n \ln(\beta) - (\alpha+1)\overbrace{\ln\left(\prod_{i=1}^{n} y_i\right)}^{\sum_{i=1}^{n}\ln(y_i)}$$

$$\ell'(\alpha) = \frac{n}{\alpha} + n\ln(\beta) - \sum_{i=1}^{n}\ln(y_i) \overset{\text{set}}{=} 0$$

$$\frac{n}{\alpha} = \sum_{i=1}^{n}\ln(y_i) - n\ln(\beta)$$

$$\boxed{\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\ln(y_i) - n\ln(\beta)}}$$

so ugly! :)

Prove it's a maximum:

$$\ell''(\alpha) = \frac{-n}{\alpha^2} < 0 \text{ always since } \alpha^2 \geq 0,\ n > 0$$