STA260 Tutorial 9 Question 3

Question 3

Let $Y_1, \ldots, Y_n$ be a random sample with the following common probability mass function:

$$f(y) = \begin{cases} \theta(1-\theta)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Here, the unknown parameter $0 < \theta < 1$.

(a) Find the MOM of θ.

(b) Find the MLE of θ.

(c) Find the MLE of $\mathbb{E}(Y_1)$.

a) $\mathbb{E}(Y) = \theta(1-\theta)^{1-1} + 2\theta(1-\theta)^{2-1} + 3\theta(1-\theta)^{3-1} + \ldots$

$= \theta + 2\theta(1-\theta) + 3\theta(1-\theta)^2 + \ldots$

$= \theta \sum_{i=1}^{\infty} i(1-\theta)^{i-1}$

Recall: $\frac{d}{dx} x^n = nx^{n-1} \Rightarrow \frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right) = \sum_{n=1}^{\infty} nx^{n-1}$

$= \theta \frac{d}{d(1-\theta)}\left(\sum_{i=1}^{\infty}(1-\theta)^i\right)$

Recall: $\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}$ if $|r|<1$. Since $0<\theta<1$,

then $0<1-\theta<1$ as well. Hence, $r = 1-\theta$ and $a=1$:

$\sum_{i=1}^{\infty}(1-\theta)^i = \frac{1-\theta}{\theta}$

$= \theta \frac{d}{d(1-\theta)}\left(\frac{1-\theta}{\theta}\right)$ for ease, let $x = 1-\theta$

$\rightsquigarrow \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{(1)(1-x) - x(-1)}{(1-x)^2} = \frac{1-x+x}{(1-x)^2} = \frac{1}{(1-x)^2}$

$\frac{1}{(1-x)^2} = \frac{1}{(1-1+\theta)^2} = \frac{1}{\theta^2}$

$= \frac{\theta}{\theta^2} = \frac{1}{\theta}$

Then, $\mathbb{E}(Y) \overset{set}{=} \bar{Y} \Rightarrow 1/\theta = \bar{Y} \Rightarrow \boxed{\hat{\theta} = 1/\bar{Y}}$

b) support does not depend on θ. Use calculus method!

$L(y_1, \ldots, y_n | \theta) = \theta(1-\theta)^{y_1-1} \times \ldots \times \theta(1-\theta)^{y_n-1}$

$= \theta^n (1-\theta)^{\Sigma(y_i-1)} = \theta^n (1-\theta)^{\Sigma y_i - n}$

$\ell(\theta) = n\ln(\theta) + (\Sigma y_i - n)\ln(1-\theta)$

$\ell'(\theta) = n/\theta - \frac{(\Sigma y_i - n)}{1-\theta} \overset{set}{=} 0$

$\Rightarrow n(1-\theta) = (\Sigma y_i - n)\theta$

$n - \cancel{n\theta} = \Sigma y_i \theta - \cancel{n\theta}$

$\theta = \frac{n}{\Sigma y_i} = \frac{1}{\bar{y}}$

Prove it's a maximum:

$\ell''(\theta) = \frac{-n}{\theta^2} - \frac{(\Sigma y_i - n)}{(1-\theta)^2}$ it is clear that $\frac{-n}{\theta^2} < 0$.

note: $\Sigma y_i = n\bar{y}$ and so $\Sigma y_i - n = n\bar{y} - n = n(\bar{y}-1)$

since $y = 1, 2, \ldots$ $\bar{y} \geq 1$. Thus $-\frac{(\Sigma y_i - n)}{(1-\theta)^2} < 0$ as well.

c) Recall: invariance property of MLE:

if $\hat{\theta}$ is the MLE for θ and any injective function $g(x)$, the MLE of $g(\theta)$ is $g(\hat{\theta})$.

Thus since $E(Y_1) = 1/\theta$ and $1/\theta$ is injective for $0<\theta<1$, and $\hat{\theta} = 1/\bar{Y}$, by the invariance property $\frac{1}{1/\bar{Y}} = \bar{Y}$ is the MLE of $E(Y_1)$.