Relevant Review from STA256

Taylor Series and Gamma Properties:

$$\sum_{k=j}^{\infty} a^k = \frac{a^j}{1-a} \text{ if } |a| \leq 1, \qquad \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x, \qquad \Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1} e^{-y} dy,$$

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \qquad \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N}$$

PDF $\Leftrightarrow$ CDF

Given that X is a continuous R.V: $\dfrac{\partial F_X(x)}{\partial x} = f_X(x), \qquad F_X(x) = \displaystyle\int_{-\infty}^{x} f_X(t)dt,$

Expected Value:

$$E(X) = \sum_x x p_x(x) \text{ or } \int_{-\infty}^{\infty} x f_x(x)dx, \qquad E(aX \pm b) = aE(X) \pm b,$$

$$E(g(X)) = \sum_x g(x)p_x(x) \text{ or } \int_{-\infty}^{\infty} g(x) f_X(x)dx, \qquad E(X) = E(E(X|Y)),$$

Variance:

$$Var(X) = E[(X-\mu)^2] = E(X^2) - [E(X)]^2, \qquad V(aX \pm b) = a^2 V(X),$$

$$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X,Y)$$

MGFs:

$$M_X(t) = E(e^{Xt}), \qquad M_X^{(k)}(0) = E(X^k)$$

Some Relevant Relationships Between Distributions Assume Y_i are identically independently distributed (i.i.d), $i = 1, 2, ..., n$, and Y_i ~Normal(μ, σ^2), then:

1. $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$
2. $Z = \frac{\overline{Y}-\mu}{\sigma/\sqrt{n}} \sim$ Normal$(0,1)$
3. $\sum_{i=1}^{n}\left(\frac{Y_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$
4. $\chi^2_{(n)} =$ Gamma$(n/2, 2)$
5. Exp$(\beta) =$ Gamma$(\alpha = 1, \beta)$
6. (Normal$(0,1))^2 = \chi^2_{(1)} =$ Gamma$(1/2, 2)$

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Question 1 (7.20 from the textbook)

(a) If U has a χ^2 distribution with v degrees of freedom, find $\mathbb{E}(U)$ and $\mathbb{V}(U)$.

(b) Using the results of Theorem 7.3, find $\mathbb{E}(S^2)$ and $\mathbb{V}(S^2)$ when $Y_1, Y_2, ..., Y_n$ is a random sample from a normal distribution with mean μ and variance σ^2. Note that S is defined as:

$$S^2 = \frac{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}{n-1}$$

Question 2

Let $Y_1, Y_2, ..., Y_n$ be n independent observations, but they're **not necessarily identically distributed**. This means it's possible some are from different distributions (normal, exponential, etc.) However, they do conveniently all have the same mean μ and finite variance σ^2. The sample variance is defined as:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

(a) Prove $\mathbb{E}[S^2] = \sigma^2$

(b) Suppose we learned that $Y_1, Y_2, ..., Y_n$ is identically distributed, and they are in fact observations from a **normal** distribution. How convenient! Find the constant a such that: $\mathbb{P}\left(\frac{3S^2}{\sigma^2} \geq a\right) = 0.9$

Question 3

Let Y_1, Y_2, Y_3, Y_4 be a random sample of size 4 from a normal population with mean 0 and variance 9. Let $\bar{Y} = \frac{1}{4}\sum_{i=1}^{n} Y_i$. Find the distribution of the following random variables.

(a) $\frac{Y_1^2}{9}$

(b) $\sum_{i=1}^{4} \frac{Y_i^2}{9}$

(c) $\sum_{i=1}^{4} \frac{(Y_i - \bar{Y})^2}{9}$

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1};\ \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta}e^{-y/\beta};\quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1}e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1}e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$; $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1}$; $y = 1, 2, \ldots$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$	$\dfrac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \dfrac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$; $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\dfrac{nr}{N}$	$n\left(\dfrac{r}{N}\right)\left(\dfrac{N-r}{N}\right)\left(\dfrac{N-n}{N-1}\right)$	
Poisson	$p(y) = \dfrac{\lambda^y e^{-\lambda}}{y!}$; $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$; $y = r, r+1, \ldots$	$\dfrac{r}{p}$	$\dfrac{r(1-p)}{p^2}$	$\left[\dfrac{pe^t}{1-(1-p)e^t}\right]^r$