STA260 Summer 2024 Tutorial 10 (10.1, 10.2, 10.10)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Relevant Review from Lecture: New Terminologies

Type I Error / False Negative: $\alpha = \mathbb{P}(\text{reject } H_0 | H_0 \text{ is true})$
Type II Error / False Positive: $\beta = \mathbb{P}(\text{accept } H_0 | H_0 \text{ is false})$
Power$(\theta) = 1 - \beta = \mathbb{P}(\text{reject } H_0 | H_0 \text{ is false})$
Note that we want to maximize power, which in plain language, is the probability of correctly rejecting the null hypothesis.
Simple: refers to "=" (i.e., $H_0 : \theta_0 = 3$ is a simple hypothesis.)
Composite: refers to "$\neq$", "$>$", "$<$" (i.e., $H_0 : \theta_0 \neq 3$ is a composite hypothesis.)
Note: in STA260, H_0 should always be simple.

NP-Lemma

Suppose $y_1, ..., y_n$ are randomly sampled, and let $\theta_0, \theta_a \in \mathbb{R}$. If you have a simple null hypothesis ($H_0 : \theta = \theta_0$) and either a simple ($H_a : \theta = \theta_a$) or a composite (i.e., $H_a : \theta \neq \theta_a$) alternative hypothesis, then the **most powerful** test is determined by the Rejection Region (RR): $\frac{L(y_1,...,y_n|\theta_0)}{L(y_1,...,y_n|\theta_a)} < k$

Remark: if $H_a : \theta > \theta_a$ or $H_a : \theta < \theta_a$, then the $\frac{L(y_1,...,y_n|\theta_0)}{L(y_1,...,y_n|\theta_a)} < k$ rejection region provides the **uniformly most powerful test.**

Question 1

Let $Y_1, \ldots, Y_8$ be a random sample from the probability density function given by:

$$f(x|\beta) = \begin{cases} \frac{3}{\beta} e^{-y^3/\beta} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the Rejection Region (RR) for the Uniformly Most Powerful (UMP) test of:

$$H_0 : \beta = 2 \quad \text{v.s.} \quad H_a : \beta > 2$$

with significance level $\alpha = 0.05$?

Hint:

$$\sum_{i=1}^{8} Y_i^3 \sim Gamma(8, 2)$$

Question 2

Let $X_1, X_2, ..., X_n$ be independent random variables such that each X_i has a $N(0, \sigma^2)$ where the variance σ^2 is unknown.

If $n = 15$, find the most powerful level $\alpha = 0.05$ test of $H_0 : \sigma^2 = 9$ vs. $H_a : \sigma^2 = 25$. Explicitly provide the Rejection Region (RR).

Question 3

For some reason a lot of people like to assume measurement error is normally distributed with a mean value μ and a standard deviation 4. Consider testing $H_0 : \mu = 0$ versus $H_a : \mu \neq 0$ based on $n = 16$ measurements.
Find the likelihood ratio test, λ, and its Rejection Region (RR) when $\alpha = 0.05$.

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1};\ \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta}e^{-y/\beta};\quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1}e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1}e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$