STA260 Summer 2024 Tutorial 12 (Final Exam Review)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Question 1

Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a random sample from a joint distribution $F_{X,Y}(x, y)$. Is $\hat{\sigma}_{X,Y} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X}_n)Y_i$ a consistent estimator of $Cov(X_i, Y_i)$?

Question 2

Let $X_1, ..., X_n$ be a random sample from a Normal(μ, σ^2) distribution.
Prove that $F = \frac{n(\bar{X}-\mu)^2}{S^2} \sim F(1, n-1)$

Question 3

Let $Y_i = x_i + E_i$ for $i = 1, ..., n$ where:

- $x_1, ..., x_n$ are fixed known constants.
- $E_1, ...E_n$ are i.i.d Normal$(0, \sigma^2)$ random variables, but σ^2 is unknown.
- Only $x_1, ..., x_n$ and $Y_1, ..., Y_n$ are observable.

(a) What is the distribution of Y_i?

(b) Find the MLE of σ^2. (Don't need to compute the second derivative test.)

(c) Is the estimator $\hat{\sigma}^2$ sufficient?

Question 4

Let $X_1, X_2, ..., X_n$ be a random sample of size n from a Gamma distribution with mean $\alpha\theta$ and variance $\alpha\theta^2$. Use the method of moments to find estimates of α and θ.

Question 5

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from an exponential distribution with mean β where $0 < \beta < \infty$.

1. Find the MLE of β and use the second derivative test to prove it's a maximum.

2. Find the MLE of $P(Y \leq 10)$.

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1};\ \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta};\quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$; $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1}$; $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$; $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$; $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$; $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$