STA260 Summer 2024 Tutorial 2 (7.3, 7.5)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Relevant Review from Lecture

Central Limit Theorem (Theorem 7.4 from textbook)

Let $Y_1, Y_2, ..., Y_n$ be independent and identically distributed random variables with $E(Y_i) = \mu$ and $V(Y_i) = \sigma^2 < \infty$. Define

$$U_n = \frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma\sqrt{n}} = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

where $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$

Then the distribution function of U_n converges to the standard normal distribution function as $n \to \infty$. That is,

$$\lim_{n\to\infty} P(U_n \le \mu) = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$

for all u.

Normal Approximation to the Binomial Distribution:

The assumptions required to perform this approximation are:

- The data is randomly sampled and responses are independent from each other.
- The population follows a binomial distribution (binary options, such as yes/no.)
- $np_e \ge 5, n(1 - p_e) \ge 5$ where n represents the sample and p_e represents the expected proportion. (This is what needs to be verified on a test.)

Continuity Correction:

- $\mathbb{P}(Y = b) = \mathbb{P}(b - \frac{1}{2} \le Y \le b + \frac{1}{2})$
- $\mathbb{P}(Y \le b) = \mathbb{P}(Y \le b + \frac{1}{2})$
- $\mathbb{P}(Y \ge b) = \mathbb{P}(Y \ge b - \frac{1}{2})$

Question 1

Let $\bar{Y}$ and S^2 be the mean and the variance of a random sample of size 25 from $N(\mu = 3, \sigma^2 = 100)$. Find $P((1 < \bar{Y} < 5) \cap (65.24 < S^2 < 189.82))$.

Hint: recall the following facts:

1. $\bar{Y}$ and S^2 are independent.
2. $\bar{Y} \sim N(\mu, \sigma^2/n)$.
3. $(n-1)S^2/\sigma^2 \sim \chi^2(n-1)$

Question 2 (7.58 from the textbook)

Suppose that $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are independent random samples from populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Show that the random variable:

$$U_n = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{(\sigma_1^2 + \sigma_2^2)/n}}$$

satisfies the conditions of Theorem 7.4 and thus that the distribution of U_n converges to a standard normal distribution function as $n \to \infty$.

Hint: Consider $W_i = X_i - Y_i$ for $i = 1, 2, ..., n$.

Question 3

Let X and Y be two independent exponential random variables with mean 1. Show that $\frac{X}{Y}$ has an F distribution and find its degrees of freedom.
Hint: First prove $Exp(1) = Gamma(1, 1)$.

Question 4

Luai is doing research to see students' perceptions of course based projects for STA304 and STA305. Approximately 66% of students claim they preferred courses with a capstone project over courses that are examination heavy. We then randomly sample 100 students. What is the probability that more than 50 of them enjoy course based projects? **Hint:** use the normal approximation to the binomial distribution, and check all assumptions to justify using the normal approximation!

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y - \mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1 - y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha + \beta}$	$\frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$