STA260 Summer 2024 Tutorial 3 (8.2, 8.3)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Relevant Review from Lecture

The point estimator $\hat{\theta}$ is called an **unbiased estimator** for a parameter θ if

$$E[\hat{\theta}] = \theta$$

We can compute the **bias** via:

$$B[\hat{\theta}] = E[\hat{\theta}] - \theta$$

The **mean square error** of a point estimator is:

$$MSE[\hat{\theta}] = E[(\hat{\theta} - \theta)^2] = V[\hat{\theta}] + [B(\hat{\theta})]^2$$

The **error of estimation** is represented by:

$$\epsilon = |\hat{\theta} - \theta|$$

The **standard error** of $\hat{\theta}$ is the same thing as the **standard deviation** of $\hat{\theta}$.

Question 1 (EZ WARM-UP!!!)

Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a population with mean μ. Show that $\sum_{i=1}^{n} a_i Y_i$ is an unbiased estimator of μ for any set of fixed constants $a_1, a_2, ..., a_n$ satisfying the condition $\sum_{i=1}^{n} a_i = 1$.

Question 2

Let $X_1, X_2, ..., X_n$ denote a random sample of size n from a distribution with the probability density function:

$$f(x) = \begin{cases} e^{-(x-\mu)} & x \geq \mu \\ 0 & \text{otherwise} \end{cases}$$

Let $X_{(1)} = \min\{X_1, X_2, ..., X_n\}$ be an estimator for μ. Calculate $B(X_{(1)})$.

Question 3

Let $Y_1, Y_2, ..., Y_n$ be a random sample from a normal distribution with mean μ and variance $\sigma^2 > 0$ and let us define the sample variance as follows:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

Prove that the estimator

$$\hat{\sigma} = \sqrt{\frac{n-1}{2}}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}S$$

is unbiased for estimating σ, where S is the square root of the sample variance.

Hint: prove that if $Y \sim Gamma(\alpha, \beta)$ then if $\alpha + a > 0$, then

$$E[Y^a] = \frac{\beta^a \Gamma(\alpha + a)}{\Gamma(\alpha)}$$

Then recall the relationship between the χ^2 and $Gamma$ distributions.

Question 4 (8.20 from the textbook)

Suppose that Y_1, Y_2, Y_3, Y_4 denote a random sample of size 4 from a population with an exponential distribution whose density is given by:

$$f(y) = \begin{cases} \frac{1}{\theta} e^{\frac{-y}{\theta}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $X = \sqrt{Y_1 Y_2}$. Find a multiple of X that is an unbiased estimator for θ.

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta}e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1}e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1}e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r,$ $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$