STA260 Summer 2024 Tutorial 4 (8.4, 8.5, 8.6)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Relevant Review from Lecture

A $100(1-\alpha)\%$ two-sided confidence interval for a parameter θ is a random interval $[\hat{\theta}_L, \hat{\theta}_U]$ such that $\mathbb{P}(\hat{\theta}_L \leq \theta \leq \hat{\theta}_U) = 1 - \alpha$.
For a lower one-sided CI, the interval is $[\hat{\theta}_L, \infty)$ and the upper one-sided CI is $(-\infty, \hat{\theta}_U]$.

The higher the $100(1-\alpha)\%$ confidence level, the more strongly believe the true parameter being estimated lies in the interval. However, this causes the interval to be more wide. If I told you I am 99% confident the weather tomorrow will be between $[10°C, 50°C]$ you would call this prediction useless!

Remark: On a test, do not say there's a $100(1-\alpha)\%$ *probability* the parameter is within the confidence interval. You need to clarify that you are $100(1-\alpha)\%$ *certain*. STA260 deals with a frequentist philosophy (not bayesian). The probability that the parameter is within an interval is either 100% or 0%; it's either contained or not.
The probability that the number five is contained in $[1, 10]$ is 100%. Meanwhile, the probability that zero is contained $[1, 10]$ is 0%.

Pivot: a random variable that is a function of the estimator θ such that it depends on $Y_1, ..., Y_n$ and its probability distribution does not depend on any unknown parameter. Finding a pivot is difficult and the instructor will not ask you to find one.

Question 1 (8.44 from the textbook)

Let Y have probability density function

$$f_Y(t) = \begin{cases} \dfrac{2(\theta - t)}{\theta^2} & 0 < t < \theta \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Show that Y has a distribution function

$$F_Y(y) = \begin{cases} 0 & y \leq 0 \\ \dfrac{2y}{\theta} - \dfrac{y^2}{\theta^2} & 0 < y < \theta \\ 1 & y \geq \theta \end{cases}$$

(b) Show that $\frac{Y}{\theta}$ is a pivotal quantity.

(c) Use the pivotal quantity from part (b) to find a 90% lower confidence limit for θ.

Question 2

Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where both parameters μ and σ^2 are unknown. A confidence interval for σ^2 can be found as follows.
We know that $(n-1)S^2/\sigma^2$ is a random variable with a $\chi^2(n-1)$ distribution. Thus we can find constants a and b so that $P((n-1)S^2/\sigma^2 < b) = 0.975$ and $P(a < (n-1)S^2/\sigma^2 < b) = 0.95$.

(a) Show that this second probability statement can be written as:

$$P\left((n-1)S^2/b < \sigma^2 < (n-1)S^2/a\right) = 0.95.$$

(b) If $n = 9$ and $s^2 = 7.93$, find a 95% confidence interval for σ^2.

Question 3

Let X be a single observation from the following probability density function:

$$f_X(x, \theta) = \frac{2x}{\theta^2}, \quad 0 \leq x \leq \theta, \quad \theta > 0$$

Construct a two-sided $(1-\alpha)100\%$ confidence interval for the parameter θ using the pivotal quantity $U = \frac{X}{\theta}$, where the probability density function of U is:

$$f_U(u) = 2u, \quad 0 \leq u \leq 1$$

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$