STA260 Summer 2024 Tutorial 5 (8.8, 8.9, Supplementary)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Relevant Review from Lecture

***F* Distribution** (Definition 7.3 from textbook)

Let W_1 and W_2 be *independent* χ^2-distributed random variables with v_1 and v_2 df, respectively. Then

$$F = \frac{W_1/v_1}{W_2/v_2}$$

is said to have an *F* distribution with v_1 numerator degrees of freedom and v_2 denominator degrees of freedom.

Most of the other necessary content has been discussed in previous tutorials; we are covering mostly supplementary questions!

Question 1 (8.97 from the textbook)

Suppose that S^2 is the sample variance based on a sample of size n from a normal population with unknown mean and variance.

1. Derive a $100(1-\alpha)\%$ upper confidence limit for σ^2.
2. Derive a $100(1-\alpha)\%$ lower confidence limit for σ^2.

Question 2 (8.125 from the textbook)

Suppose that independent samples of sizes n_1 and n_2 are taken from two normally distributed populations with variances σ_1^2 and σ_2^2, respectively. If S_1^2 and S_2^2 denote the respective sample variances, Theorem 7.3 implies that $(n_1-1)S_1^2/\sigma_1^2$ and $(n_2-1)S_2^2/\sigma_2^2$ have χ^2 distributions with $n_1 - 1$ and $n_2 - 1$ degrees of freedom, respectively. Further, these χ^2 distributed random variables are independent because the samples were independently taken.

1. Use these quantities to construct a random variable that has an F distribution with n_1-1 numerator degrees of freedom and n_2-1 denominator degrees of freedom.

2. Use the F-distributed quantity from part (a) as a pivotal quantity and derive a formula for a $100(1-\alpha)\%$ confidence interval for $\frac{\sigma_2^2}{\sigma_1^2}$.

Question 3 (8.129 from the textbook)

If
$$S_\star^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n} \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1}$$
then $S_\star^2$ is a biased estimator of σ^2, but S^2 is an unbiased estimator for the same parameter. If we sample from a normal population,

(a) Find $\mathbb{V}(S_\star^2)$

(b) Prove $\mathbb{V}(S^2) > \mathbb{V}(S_\star^2)$

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$