STA260 Summer 2024 Tutorial 6 (9.1, 9.2, 9.3)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Relevant Review from Lecture: Relative Efficiency

It is possible to obtain more than one unbiased estimator. You should've learned from lecture that the unbiased estimator is not necessarily always the best estimator. One way to find a better estimator is to find one with the **least variance**. We want to have one with the least variance, because intuitively variance tells you the average spread between two data points. This is the motivation for **relative efficiency**:

$$\text{eff}(\hat{\theta}_1, \hat{\theta}_2) = \frac{V(\hat{\theta}_2)}{V(\hat{\theta}_1)}$$

Where $\hat{\theta}_1, \hat{\theta}_2$ are unbiased estimators. If $\text{eff}(\hat{\theta}_1, \hat{\theta}_2) > 1$, then we can say that $\hat{\theta}_1$ is a better unbiased estimator.

Relevant Review from Lecture: Consistency

$\hat{\theta}_n$ is a consistent estimator of θ if $\forall \epsilon > 0$, $\lim_{n\to\infty} \mathbb{P}(|\hat{\theta}_n - \theta| \leq \epsilon) = 1$ or alternatively, $\lim_{n\to\infty} \mathbb{P}(|\hat{\theta}_n - \theta| > \epsilon) = 0$. In plain English, we could interpret this as we increase the sample size (n), then the probability that the error of estimation is less than a very small number (ϵ) is 1!

Theorem 9.1: an unbiased estimator $\hat{\theta}_n$ for θ is consistent if $\lim_{n\to\infty} V(\hat{\theta}_n) = 0$. This theorem is intuitive; if the variance decreases to 0, this suggests that the error of estimation is also close to 0 (which would be less than a very small arbitrary number.)

Methods for proving consistency:
1. Definition (see exercises 9.26-9.29).
2. Theorem 9.1 (easiest if the expected value and variance can be easily computed.)
3. Weak Law of Large Numbers (WLLN):

$$\text{as } n \to \infty,\ \bar{X}_n \xrightarrow{p} \mathbb{E}(X_i) \text{ (assuming } X_i\text{'s are iid.)}$$

4. Continuous mapping theorem (usually combined with WLLN):

$$\text{if } \hat{\theta}_n \xrightarrow{p} c, c \in \mathbb{R} \text{ and } g(x) \text{ is a continuous function at } y = c \text{ then } g(\hat{\theta}_n) \xrightarrow{p} g(c).$$

Question 1 (9.3 from the textbook)

If $Y_1, Y_2, ..., Y_n$ denote a random sample from the $Uniform(\theta, \theta + 1)$ distribution, then both

$$\hat{\theta}_1 = \bar{Y} - \frac{1}{2} \quad \text{and} \quad \hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}$$

are unbiased estimators for θ. Which one is the better unbiased estimator? Calculate the efficiency of $\hat{\theta}_1$ relative to $\hat{\theta}_2$.

Question 2 (9.30 from the textbook)

Let $Y_1, Y_2, ..., Y_n$ be independent random variables, each with the probability density function:

$$f(y) = \begin{cases} 3y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\bar{Y}$ converges in probability to some constant and state which exact constant.

Question 3 (9.34 from the textbook)

Let Rayleigh density function is given by:

$$f(y) = \begin{cases} \left(\frac{2y}{\theta}\right) e^{\frac{-y^2}{\theta}} & y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

From a previous exercise (6.34) it is proven that Y^2 has an exponential distribution with mean θ. If $Y_1, Y_2, ..., Y_n$ denote a random sample from a Rayleigh distribution, prove that $W_n = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$ is a consistent estimator for θ.

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \leq y \leq \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$; $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1}$; $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$; $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$; $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$; $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$