STA260 Summer 2024 Tutorial 7 (9.4, Completeness, 9.5)

Relevant Review from Lecture: Sufficiency

U is a sufficient statistic for θ if and only if: $L(y_1, ..., y_n|\theta) = g(U, \theta) \times h(y_1, ..., y_n)$. It is mandatory for $g(U, \theta)$ to be a function that contains θ (it cannot be a constant) whereas $h(y_1, ..., y_n)$ just needs to be a function without θ.

Relevant Review from Lecture: Completeness

A statistic U is complete if and only if every function $g(U)$ such that $\mathbb{E}(g(U)) = 0, \forall\theta$ implies that $g(U) = 0$ almost everywhere (a.e). There are two methods:

1. **Definition.** Use this when the support depends on θ and/or when you need to prove $Y_{(1)} = \min\{Y_1, ..., Y_n\}$ or $Y_{(n)} = \max\{Y_1, ..., Y_n\}$ is complete.
   Method: set $\mathbb{E}(g(\hat{\theta})) = \int g(y) f_{\hat{\theta}}(y) dy$ to equal 0 and then prove $g(\theta) = 0$.

2. **Exponential family.** When the support does not depend on θ, the distribution may belong to the exponential family (in fact, many common distributions do!) This also proves sufficiency, so it's highly efficient.
   Method: prove the distribution has the form: $f(y|\theta) = e^{p(\theta)k(y)+q(\theta)+s(y)}$ and therefore $U = \sum_{i=1}^{n} k(y_i)$ is sufficient and complete.

Relevant Review from Lecture: Rao-Blackwell Theorem

If $\hat{\theta}$ is an **unbiased** estimator for θ and U is a **sufficient** statistic for θ and contains $\hat{\theta}$, then $\mathbb{E}(\hat{\theta}|U) = \hat{\theta}$ is the MVUE (Minimum Variance Unbiased Estimator). Prove that U (the sufficient statistic) is also complete to prove it's the UMVUE Unique Minimum Variance Unbiased Estimator). There are two common procedures:

1. If the distribution is known, i.e., exponential, gamma, normal... Then you should:
   i. Find an unbiased estimator U_1
   ii. Find a sufficient estimator U_2
   Then $\mathbb{E}(U_1|U_2) = U_1$ is the MVUE. (For UMVUE, U_2 should be complete.)

2. If the distribution is not known, then you should:
   i. Find a sufficient estimator U
   ii. Find a function $g(x)$ such that $g(U)$ is an unbiased estimator for θ.
   Then $\mathbb{E}(g(U)|U) = g(U)$ is the MVUE. (For UMVUE, U must also be complete.)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Question 1

Let $Y_1, ..., Y_n$ be a random sample with the following probability density function:

$$f(y) = \begin{cases} \frac{y}{\theta} e^{\frac{-y^2}{2\theta}} & y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Where $\theta > 0$. Find the sufficient statistic using the factorization theorem, and provide $g(u, \theta)$ and $h(y_1, ..., y_n)$.

Question 2

Let $Y_1, ..., Y_n$ be a random sample from a population density function:

$$f(y) = \begin{cases} \frac{3y^2}{\theta^3} & 0 \leq y \leq \theta \\ 0 & \text{otherwise.} \end{cases}$$

Show that $Y_{(n)} = \max\{Y_1, ..., Y_n\}$ is complete.

Question 3

Let $Y_1, ..., Y_n$ be a random sample from a $Bernoulli(p)$ distribution. Find the MVUE of $(1-p)^2$.

Question 4

Let $Y_1, ..., Y_n$ be a random sample from a $Gamma(2, \beta)$ distribution. Find the UMVUE of $\beta(\beta + 2)$.

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta}e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1}e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1}e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$; $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1}$; $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$; $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$; $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$; $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$