STA260 Summer 2024 Tutorial 8 (Cramer-Rao Inequality, Exponential Family)

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Recall: Exponential Family

If the distribution has the form: $f(y|\theta) = e^{p(\theta)k(y)+q(\theta)+s(y)}$ then it is part of the exponential family. Additionally, $U = \sum_{i=1}^{n} k(y_i)$ is sufficient and complete.

Relevant Review from Lecture: Cramer-Rao Inequality

If $\hat{\theta}$ is unbiased and $V(\hat{\theta}) = \frac{1}{I_n(\theta)}$ then $\hat{\theta}$ is the MVUE.

$$I_n(\theta) = nI(\theta) = n\left(-\left(\mathbb{E}\left[\left(\frac{d^2}{d^2\theta}\ln(f_Y(y|\theta))\right)\right]\right)\right)$$

Remark: this may only be used if the support is free from θ. It is also not recommended compared to Rao-Blackwell unless $V(\hat{\theta})$ can be easily computed.

Question 1

Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a population with common probability mass function

$$p_Y(y \mid \theta) = \theta^y (1-\theta)^{1-y}, \quad y = 0 \text{ or } 1,$$

where $0 \leq \theta \leq 1$ is a parameter.

(a) Derive the Fisher information $I_n(\theta)$ of this distribution.

(b) Prove that $\bar{Y}$ is MVUE of θ using the Cramer-Rao theorem.

Question 2

Let $Y_1, ..., Y_n$ denote a random sample from $N(0, \theta^2)$ where $\theta > 0$ is unknown. Compute the Cramer-Rao Lower Bound.

Question 3

Let $Y_1, ..., Y_n$ be a random sample with the following common probability density function:

$$f(y) = \begin{cases} \frac{1}{\theta^2} y e^{\frac{-y}{\theta}} & y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Here $\theta > 0$. Prove that $\sum_{i=1}^{n} Y_i$ is a complete sufficient statistic for θ.

Question 4

Let $Y_1, ..., Y_n$ be a random sample with the following common probability density function:

$$f(y) = \begin{cases} \frac{1}{2}e^{\frac{-(y-\theta)}{2}} & y > \theta \\ 0 & \text{otherwise} \end{cases}$$

Here, $\theta \in \mathbb{R}$. Determine whether $f(y)$ is part of the exponential family.

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \leq y \leq \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$; $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1}$; $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}}$; $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$; $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$; $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$