Relevant Review from Lecture: Maximum Likelihood Method

Let $Y_1, Y_2, ..., Y_n \overset{i.i.d}{\sim} f(y|\theta)$. Then, let:

$$\ell(\theta) = \ln(L(y_1, ..., y_n|\theta)) = \ln(\Pi_{i=1}^n f(y_i|\theta))$$

$\hat{\theta}$ is known as the MLE (maximum likelihood estimator) if and only if $\hat{\theta}$ maximizes $\ell(\theta)$. There are two ways of finding the MLE:

1. $\ell(\theta)$ is differentiable, which commonly occurs when the support does not contain the parameter of interest θ.

   i Calculate $\ell'(\theta)$ then set it to equal 0. Solve for θ.

   ii Prove $\ell''(\hat{\theta}) < 0$ to prove that it is a maximum.

2. $\ell(\theta)$ is not differentiable, which commonly occurs when the support contains the parameter of interest θ. You can typically draw graphs and make conclusions. In general, there's the following trend:

   i If the support is $y > \theta$, then typically the MLE is $Y_{(1)} = \min\{Y_1, ..., Y_n\}$.

   ii If the support is $y < \theta$, then typically the MLE is $Y_{(n)} = \max\{Y_1, ..., Y_n\}$.

Invariance Property: if $\hat{\theta}$ is the MLE for θ, then for any function $g(x)$ where the permissible values of θ guarantees that $g(x)$ is injective, then the MLE of $g(\theta)$ is $g(\hat{\theta})$.

Important notes: the MLE is **not unique** and doesn't necessarily mean it is unbiased. The fact $\sum_{i=1}^n y = n\bar{y}$ is useful for proving that $\ell''(\hat{\theta}) < 0$.

Relevant Review from Lecture: Method of Moments Method

The k^{th} moment of a random variable is $\mathbb{E}(y^k)$. The k^{th} sample moment is $\frac{1}{n}\sum_{i=1}^n y_i^k$. The strategy is to set $\mathbb{E}(y) = \bar{y}$ and to solve for θ. If $\mathbb{E}(y)$ is not a function θ, then try higher orders, such as: $\mathbb{E}(y^2) = \frac{1}{n}\sum_{i=1}^n y_i^2$, $\mathbb{E}(y^3) = \frac{1}{n}\sum_{i=1}^n y_i^3$, etc.

Disclaimer: STA260 covers extensive material. Tutorials serve as additional aids, but they cannot cover every question type.

Question 1

Let $Y_1, Y_2, ..., Y_n$ denote a random sample from a distribution with the following probability density function with parameters $\alpha > 0$ and $\beta > 0$, where β is known.

$$f(y|\alpha, \beta) = \begin{cases} \alpha\beta^\alpha y^{-(\alpha+1)} & y \geq \beta \\ 0 & \text{otherwise} \end{cases}$$

Find the MLE for α.

Question 2

Consider a random sample $Y_1, Y_2, ..., Y_n$ from the following probability density function:

$$f(y|\alpha) = \begin{cases} \frac{\alpha 2^{\alpha}}{y^{\alpha+1}} & y > 2, \quad \alpha > 0 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, assume $\alpha \neq 1$. Derive the method of moments estimator for α.

Question 3

Let $Y_1, ..., Y_n$ be a random sample with the following common probability mass function:

$$f(y) = \begin{cases} \theta(1-\theta)^{y-1} & y = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Here, the unknown parameter $0 < \theta < 1$.

(a) Find the MOM of θ.

(b) Find the MLE of θ.

(c) Find the MLE of $\mathbb{E}(Y_1)$.

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1}; \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta}; \quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$