Question 1 (Section 1 Question 4)

John, Jim, Jay, and Jack have formed a band consisting of 4 instruments.

(a) If each of the boys can play all 4 instruments, how many different arrangements are possible?

(b) What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?

Solution

(a) For instrument 1, we can pick one of the 4 different band members. Hence, there are 4 possibilities.
Now, for instrument 2, we have 3 members remaining, so 3 possibilities.
Again, for instrument 3 we have 2 members remaining, so 2 possibilities.
And then the last member (that wasn't picked) must play instrument 4.
Hence we have: $4 \times 3 \times 2 \times 1 = 4! = 24$ different arrangements. Notice that we multiply to account for all the different combinations of possibilities. (This can be visualized using a tree diagram if you are not convinced.)

(b) Let the piano represent the first instrument and the drums represent the second instrument. If Jay and Jack can only play the piano and drums, then we shouldn't assign John or Jim to these instruments (or else all 4 instruments will not be played and their band fails!)
Hence for instrument 1 we have 2 possibilities. Instrument 2 has 1 possibility. Now, it can either be the case that John or Jim play the third instrument and then the other must play the fourth.
Hence for instrument 3 we have 2 possibilities. Instrument 4 has 1 possibility. Hence we have: $2 \times 1 \times 2 \times 1 = (2!)(2!) = 2(2) = 4$ different arrangements.

Question 2 (Section 1 Question 7)

(a) In how many ways can 3 boys and 3 girls sit in a row?

(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?

(c) In how many ways if only the boys must sit together?

(d) In how many ways if no two people of the same sex are allowed to sit together?

Solution

(a) We have 6 people in total, all needing to be aligned in a row.
For the first person in the row, there are 6 choices.
For the second person in the row, there are 5 choices.
...
For the last person, there is only 1 choice left.
Hence we have $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$ different ways.
For the below solutions we will assume the audience now has a better understanding of permutations.

(b) There are $3!$ ways to rearrange the order of the boys, $3!$ ways to rearrange the order of the girls, and then $2!$ ways to decide whether boys come first or girls come first.
Hence we have: $3! \times 3! \times 2! = 72$

(c) Again, there are $3!$ ways to arrange the boys. Since they all must be together, but the girls don't need to, we know that the boys can be in-between some girls. Hence, for the first placement in the row there are 4 options: either girl 1, girl 2, girl 3, or the boys. Similar to the other cases, we have $4!$ ways of rearranging them.
Hence we have: $3! \times 4! = 144$

(d) If this happens, notice we have one of two cases:
girl-boy-girl-boy-girl-boy
boy-girl-boy-girl-boy-girl
Hence we can see that the arrangements of the boys or the girls are pre-determined (i.e., we know that there are $3!$ ways to choose the placement for the boys, and similarly for the girls).
Hence the number of different arrangements are: $2 \times 3! \times 3! = 72$

Question 3 (Section 1 Question 17)

A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

Solution

We must choose 5 women from the group of 10 and 5 men from the group of 12. The order doesn't matter; hence we can use combinations:

$$\binom{10}{5}\binom{12}{5}$$

Now we need to assign the pairs. Assume that the order of the pair doesn't matter (i.e., pair [woman 1, man 1] is the same as [man 1, woman 1]), then we can just focus on rearranging the men to women. There are 5 men for woman 1, then 4 men for woman 2, etc... Hence we have $5!$ ways to rearrange the pairs. So the resulting answer is:

$$\binom{10}{5}\binom{12}{5}5! = 23950080$$

Question 4 (Section 1 Question 18)

Student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if:

(a) Both books are to be on the same subject?

(b) The books are to be on different subjects?

Solution

(a) First, we must pick a subject (out of 3: math, science, or economics). Then, we need to choose 2 from the same subject. Note that the cases where the textbooks are math, science, or economics are disjoint. Hence we need to add: num. of ways to pick 2 math + num. of ways to pick 2 science + num. of ways to pick 2 economics. This results in:

$$\binom{6}{2} + \binom{7}{2} + \binom{4}{2} = 42$$

(b) Here we can either have one of the following pairs: (math, science), (math, economics), or (science, economics).
(Fun fact: if you want to generalise this problem to n different subjects, note that we could compute the number of pairs through:

$$\binom{n}{2}$$

In this case, since $n = 3$, there were 3 pairs.)
So now we just need to add for the three different pairs. This results in:

$$\binom{6}{1}\binom{7}{1} + \binom{6}{1}\binom{4}{1} + \binom{7}{1}\binom{4}{1} = 94$$

An **alternative** solution can be solved as follows: we know from part a) that the number of ways such that two of them must be the same subject is 42. We can instead count the total number of cases to pick 2 books from all different subjects and subtract the cases where two of them are the same (this results in having two books of different subjects.)
Since we have $6+7+4 = 17$ different books to choose from, the answer then would be:

$$\binom{17}{2} - 42 = 94$$