Question 1 (Section 2 Question 5)

A system is composed of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_4)$ where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

(a) How many outcomes are in the sample space of this experiment?

(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

(c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

(d) Write out all the outcomes in the event AW.

Solution

(a) For each component, there are two outcomes: either it is working or it is failed. And since we have five of them, we then have $2^5 = 32$ different outcomes.

(b) Let's consider the following cases:

- Both components 1 and 2 are working (vector representation is $(1, 1, x_3, x_4, x_5)$), and we don't care whether the remaining three components work. Hence, there are: $1^2 \times 2^3 = 8$ different outcomes.
- Both 3 and 4 are working and we don't whether the remaining three components work (vector representation is $(x_1, x_2, 1, 1, x_5)$). Similar to the last case, we have 8 different outcomes. However, we need to account for overlap, i.e., in the precious part we accounted for the case of all components $(1, 1, 1, 1, 1)$ or just four of them $(1, 1, 1, 1, x_5)$ to work. Hence this case only provides 6 additional outcomes.
- Both components $1, 3, 5$ work and we don't care whether the remaining two components work (vector representation is $(1, x_2, 1, x_4, 1)$). Hence we have $1^3 \times 2^2 = 4$ different outcomes. However, there is again overlap. For ease of explanation, notice the following overlap:

$$\{(1, 0, 1, 0, 1), (1, 1, 1, 0, 1), (1, 0, 1, 1, 1), (1, 1, 1, 1, 1)\}$$

  The last 3 cases were already accounted for the above, so we can only add 1 additional outcome.

Hence we have 15 unique outcomes in W.

(c) We can simply represent $A = \{(x_1, x_2, x_3, 0, 0) : x_i \in (0, 1)\}$. Hence we have $2^3 \times 1^2 = 8$ unique outcomes.

(d) If it must be the case that components 4 and 5 fail but the system must work, then it must be the case that components 1 and 2 are working (as the other requirements involve component 4 or 5). Hence: $AW = \{(1, 1, x_3, 0, 0)\}$ which is of size 2.

Question 2 (Section 2 Question 10)

Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing

(a) a ring or a necklace?

(b) a ring and a necklace?

This solution can be simplified by using a table. Notice that the question provides the following information:

		Wear Ring		
		No	Yes	
Wear Necklace	No	60%		
	Yes			30%
			20%	100%

Hence filling out the table gives:

		Wear Ring		
		No	Yes	
Wear Necklace	No	60%	10%	70%
	Yes	20%	10%	30%
		80%	20%	100%

Solution

Using the above table, the question is now trivial.

(a) From the table, we can add the following disjoint cases: the person wearing a ring (but not a necklace), the person wearing a necklace (but not a ring), or wearing both. This results in $10\% + 20\% + 10\% = 40\%$.

(b) From the table, it's just 10%.

Question 3 (Section 2 Question 15)

If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt

(a) a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)

(b) one pair? (This occurs when the cards have denominations a, a, b, c, d where a, b, c, d are all distinct.)

(c) two pairs? (This occurs when the cards have denominations a, a, b, b, c where a, b, c are all distinct.)

(d) three of a kind? (This occurs when the cards have denominations a, a, a, b, c where a, b, c and are all distinct.)

(e) four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

Solution (Part a, b)

Note that there are four suits in poker, and each suit contains 13 cards or "denominations". Here, the denominations refer to ace, numbers 2 to 10, jack, queen, and king.

(a) First, we need to pick one of the four suits. Then, we just need to pick five out of the thirteen cards. Hence the probability is:

$$\binom{4}{1}\binom{13}{5} \div \binom{52}{5} = 0.001980792 \approx 0.002$$

(b) There are 13 denominations and we want to choose one to make our pair $\binom{13}{1}$. Then, we need to pick two suits in each the denominations belong to $\binom{4}{2}$. Now, we need to pick 3 denominations that are not identical to the one made for the pair $\binom{12}{3}$. Finally, we need to pick the suit said the non-pairs belong to $\binom{4}{1}$. And we have three remaining cards, so we must cube the last expression. This results in:

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 \div \binom{52}{5} = 0.422569 \approx 0.4226$$

Solution (Parts c-e)

(c) There are 13 denominations and we want to choose two to make our two pairs $\binom{13}{2}$. Then, we need to pick two suits in each the denominations belong to, for both pairs $\binom{4}{2}^2$. Now, we need to pick 1 denomination that are not identical to the ones belonging to the pairs $\binom{11}{1}$. Finally, we need to pick the suit for the remaining card $\binom{4}{1}$. This results in:

$$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} \div \binom{52}{5} = 0.04753902 \approx 0.0475$$

(d) There are 13 denominations and we want to choose 1 where there will be three cards of the sane denomination $\binom{13}{1}$. Then, we need to pick the suits in which these three cards belong to $\binom{4}{3}$. Now, we need to pick 2 denominations that are not identical to the ones belonging to the pairs $\binom{12}{2}$. Finally, we need to pick the suit for the remaining cards $\binom{4}{1}^2$. This results in:

$$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 \div \binom{52}{5} = 0.02112845 \approx 0.0211$$

(e) There are 13 denominations and we want to choose 1 where there will be four cards of the sane denomination $\binom{13}{1}$. Then, we need to pick the suits in which these four cards belong to $\binom{4}{4}$. Now, we need to pick 1 denomination that is different from the four-of-a-kind $\binom{12}{1}$, and we need to pick a suit $\binom{4}{1}$. This results in:

$$\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1} \div \binom{52}{5} = 0.000240096 \approx 0.0002$$

Question 4 (Section 2 Question 17)

Twenty five people, consisting of 15 women and 10 men are lined up in a random order. Find the probability that the ninth woman to appear is in position 17. That is, find the probability there are 8 women in positions 1 through 16 and a woman in position 17.

Solution

Note that the order actually **only** matters when we reach the 17th position. Hence, we must choose 8 women and 8 men to come before the 17th position. This can be chosen by:

$$\binom{15}{8}\binom{10}{8}\binom{7}{1}$$

Where $\binom{7}{1}$ comes from choosing one of the remaining 7 women to be placed. Now, the total number of ways to pick 17 people to be arranged is just:

$$\binom{25}{16}\binom{9}{1}$$

Where $\binom{25}{16}$ comes from just selecting 16 participants to begin with, and $\binom{9}{1}$ represents finding the remainder to be placed as 17th. Hence the probability is:

$$\binom{15}{8}\binom{10}{8}\binom{7}{1} \div \binom{25}{16}\binom{9}{1} = 0.1102436 \approx 0.1102$$

Notice that this question doesn't care about the rest of the arrangements, since we only care about the 17th position. (If we were to let's say count the rest, we would multiply the numerator and denominator by $\binom{8}{8}$ which is just equal to 1.)

For students who prefer to use permutations instead of combinations, the alternative answer is found below. (Warning! It's also more complicated, and would take longer to think about... Which you can see why we ended up using combinations.)

Alternative Solution

Some students will prefer permutations instead of combinations. Although this is fine, it's complicated (but we can get the same final answer actually.) First we will find the denominator (the total number of ways to rearrange men and women). First, out of all 25 participants let's fix the position of where the women are positioned $\binom{25}{15}$. Now, there are $15!$ women to rearrange and $10!$ men to rearrange. This gives us:

$$\binom{25}{15} \times 15! \times 10!$$

Now let's count the number of ways to rearrange the men and women such that there is a woman in position 17. First, we must fix the positions of where the eight women must be placed from 1 to 16. This gives us $\binom{16}{8}$. Once the positions are placed, we know that we have $15 \times 14 \times ... \times 9 \times 8 = Perm(15, 8)$ ways to place these women. Similarly, there are $Perm(10, 8)$ ways to place the men. Now, on the 17th position it must be one of the remaining 7 women $\binom{7}{1}$. Now we need to rearrange the rest. Again, fix the positions of the leftover women $\binom{8}{6}$. Then, rearrange the rest of the women $6!$ and the rest of the men 2!. This gives us with the numerator:

$$\binom{16}{8} \times Perm(15, 8) \times Perm(10, 8) \times \binom{7}{1} \times \binom{8}{6} \times 6! \times 2!$$

Hence the probability is:

$$\left[\binom{16}{8} \times Per(15, 8) \times Per(10, 8) \times \binom{7}{1} \times \binom{8}{6} \times 6! \times 2!\right] \div \left[\binom{25}{15} \times 15! \times 10!\right]$$

Which will give the same answer as the first solution.