Question 1 (Section 3 Question 9)

Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white balls and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

Solution

Here we are trying to compute:

$$\mathbb{P}(\text{Picked white ball from urn A}|\text{2 balls are white})$$

$$= \frac{\mathbb{P}(\text{Picked white ball from urn A \& 2 balls are white})}{\mathbb{P}(\text{2 balls are white})}$$

Note that for two balls to be white, it must be one of these three cases:

1. Picking a white ball from urns A & B, then a red ball in urn C. Note that the ball being picked from each urn is independent from each other, hence the probability is:

$$\mathbb{P}(\text{White from Urn A}) \times \mathbb{P}(\text{White from Urn B}) \times \mathbb{P}(\text{Red from Urn C}) = \frac{2}{6} \times \frac{8}{12} \times \frac{3}{4}$$

2. Picking a white ball from urns A & C, then a red ball in urn B. Probability:

$$\mathbb{P}(\text{White from Urn A}) \times \mathbb{P}(\text{White from Urn B}) \times \mathbb{P}(\text{Red from Urn C}) = \frac{2}{6} \times \frac{4}{12} \times \frac{1}{4}$$

3. Picking a white ball from urns B & C, then a red ball in urn A. Probability:

$$\mathbb{P}(\text{White from Urn A}) \times \mathbb{P}(\text{White from Urn B}) \times \mathbb{P}(\text{Red from Urn C}) = \frac{4}{6} \times \frac{8}{12} \times \frac{1}{4}$$

Note that cases 1 and 2 are associated with picking a white ball from urn A & 2 balls are white. Hence the answer is:

$$\frac{\left(\frac{2}{6} \times \frac{8}{12} \times \frac{3}{4}\right) + \left(\frac{2}{6} \times \frac{4}{12} \times \frac{1}{4}\right)}{\left(\frac{2}{6} \times \frac{8}{12} \times \frac{3}{4}\right) + \left(\frac{2}{6} \times \frac{4}{12} \times \frac{1}{4}\right) + \left(\frac{4}{6} \times \frac{8}{12} \times \frac{1}{4}\right)} = \frac{7}{11}$$

Question 2 (Section 3 Question 11)

Two cards are randomly chosen without replacement from an ordinary deck of 52 cards. Let B be the event that both cards are aces, let A_s be the event that the ace of spades is chosen, and let A be the event that at least one ace is chosen. Find:

(a) $\mathbb{P}(B|A_S)$

(b) $\mathbb{P}(B|A)$

Solution

(a) Note that there's only one ace of spades, so $\mathbb{P}(A_S) = \frac{1}{52}$. Now, the probability that the second card is a ace given that the second card is an ace of spades is $\mathbb{P}(B \cap A_S) = \frac{1}{52} \times \frac{3}{51}$. Hence:

$$\mathbb{P}(B|A_S) = \frac{\mathbb{P}(B \cap A_S)}{\mathbb{P}(A_S)} = \frac{\frac{1}{52} \times \frac{3}{51}}{\frac{1}{52}} = \frac{1}{17}$$

(b) Note that the probability that two aces are chosen is $\frac{\binom{4}{2}}{\binom{52}{2}}$ and the probability that only one ace is chosen is $1 - \mathbb{P}(\text{no ace}) = 1 - \frac{\binom{48}{2}}{\binom{52}{2}}$. Hence the probability is:

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\frac{\binom{4}{2}}{\binom{52}{2}}}{1 - \frac{\binom{48}{2}}{\binom{52}{2}}} = \frac{1}{33}$$

Question 3 (Section 3 Question 18)

In a certain community, 36% of the families own a dog and 22% of the families that own a dog also own a cat. In addition, 30% of the families own a cat. What is:

(a) the probability the a random select family owns both a dog and a cat?

(b) the conditional probability that a randomly selected family owns a dog given that it owns a cat?

Solution

(a) Notice that:

$$\mathbb{P}(\text{Owning a cat}|\text{Owning a dog}) = \frac{\mathbb{P}(\text{Owning a cat and a dog})}{\mathbb{P}(\text{Owning a dog})}$$

$$22\% = \frac{\mathbb{P}(\text{Owning a cat and a dog})}{36\%} \quad \Leftrightarrow \quad \mathbb{P}(\text{Owning a cat and a dog}) = 0.0792$$

(b) We can use some facts from the previous question.

$$\mathbb{P}(\text{Owning a dog}|\text{Owning a cat}) = \frac{\mathbb{P}(\text{Owning a cat and a dog})}{\mathbb{P}(\text{Owning a cat})}$$

$$\mathbb{P}(\text{Owning a dog}|\text{Owning a cat}) = \frac{0.0792}{0.3} = 0.264$$

Question 4 (Section 3 Question 32)

Consider two boxes, one containing 1 black and 1 white marble, the other 2 black and 1 white marble. A box is selected at random, and a marble is drawn from it at random.

(a) What is the probability that the marble is black?

(b) What is the probability that the first box was the one selected given that the marble is white?

Solution

(a) There are two cases to consider: if box 1 is chosen, then there's a $(1/2)$ chance of getting a black ball. If box 2 is chosen, then there's a $(2/3)$ chance of getting a black ball. And the probability of getting either box 1 or 2 is equally likely. Hence:

$$\mathbb{P}(\text{Obtaining box 1 and have a black ball}) + \mathbb{P}(\text{Obtaining box 2 and have a black ball})$$

$$\mathbb{P}(\text{black ball is drawn}) = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{7}{12}$$

(b) From the first part, the probability of getting a white ball is $1 - \frac{7}{12} = \frac{5}{12}$. Note that the probability of getting the first box then getting a white ball is $\frac{1}{2} \times \frac{1}{2}$. Hence, the answer is:

$$\mathbb{P}(\text{choosing box 1}|\text{white ball is drawn}) = \frac{\mathbb{P}(\text{choosing box 1 and a white ball})}{\mathbb{P}(\text{white ball is drawn})}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{5}{12}} = \frac{3}{5}$$

Question 5 (Section 3 Question 42)

Consider a sample of size 3 drawn in the following manner: we start with an urn containing 5 white and 7 red balls. At each stage, a ball is drawn and its colour is noted. The ball is then returned to the urn, along with an additional ball of the same colour. Find the probability that the sample will contain exactly

(a) 0 white balls

(b) 1 white ball

(c) 3 white balls

(d) 2 white balls

Solution

(a) This means the first, second, and third ball must be red. Noting that we get an extra red ball for each placement, the answer is:

$$\mathbb{P}(0 \text{ white balls}) = \frac{7}{12} \times \frac{8}{13} \times \frac{9}{14} = \frac{3}{13}$$

(b) First we choose where to place the order of the white ball $\binom{3}{1}$. Then, we know that there is one white ball, but once the red ball gets added the next entry must be red. Hence:

$$\mathbb{P}(1 \text{ white balls}) = \binom{3}{1} \times \frac{5}{12} \times \frac{7}{13} \times \frac{8}{14} = \frac{5}{13}$$

(c) This means all of the balls must be white. Noting that we get an extra white ball for each placement, the answer is:

$$\mathbb{P}(3 \text{ white balls}) = \frac{5}{12} \times \frac{6}{13} \times \frac{7}{14} = \frac{5}{52}$$

(d) First we choose the order of the white balls $\binom{3}{2}$. Then, we know that there's one extra white ball for us to pick. Hence:

$$\mathbb{P}(2 \text{ white balls}) = \binom{3}{2} \times \frac{5}{12} \times \frac{6}{13} \times \frac{7}{14} = \frac{15}{52}$$