Question 1 (Section 4 Question 1)

Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings. What are the possible values of X, and what are the probabilities associated with each value?

Solution

Note that we have 4 unique scenarios:

- We have 2 black balls and we win \$4. Probability: $\frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$
- We have 1 black ball, 1 orange ball. We win \$2. Probability: $\frac{\binom{4}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$
- We have 1 black ball, 1 white ball. We win \$1. Probability: $\frac{\binom{4}{1}\binom{8}{1}}{\binom{14}{2}} = \frac{32}{91}$
- We have 2 orange balls. Nothing happens! Probability: $\frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$
- We have 1 white ball, 1 orange ball. We lose \$1. Probability: $\frac{\binom{8}{1}\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$
- We have 2 white balls. We lose \$2. Probability: $\frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$

Hence the possible values of X are $4, 2, 1, 0, -1, -2$ and the probability mass function is as follows:

$$P(X = x) = \begin{cases} \frac{6}{91} & x = 4 \\ \frac{8}{91} & x = 2 \\ \frac{32}{91} & x = 1 \\ \frac{1}{91} & x = 0 \\ \frac{16}{91} & x = -1 \\ \frac{28}{91} & x = -2 \end{cases}$$

Question 2 (Section 4 Question 2)

Two fair dice are rolled. Let X equal the product of 2 dice. Compute $\mathbb{P}(X = i)$ for $i = 1, ..., 36$.

This question is trivial if we make a table. Consider the following sums from a die roll, where each outcome is equally as likely:

		Outcome on Die 1					
		1	2	3	4	5	6
Outcome on Die 2	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

Solution

Reading off of the table, we obtain:

- $\mathbb{P}(X = 1) = \frac{1}{36}$, $\mathbb{P}(X = 2) = \frac{2}{36}$, $\mathbb{P}(X = 3) = \frac{2}{36}$
- $\mathbb{P}(X = 4) = \frac{3}{36}$, $\mathbb{P}(X = 5) = \frac{2}{36}$, $\mathbb{P}(X = 6) = \frac{4}{36}$
- $\mathbb{P}(X = 8) = \frac{2}{36}$, $\mathbb{P}(X = 9) = \frac{1}{36}$, $\mathbb{P}(X = 10) = \frac{2}{36}$
- $\mathbb{P}(X = 12) = \frac{4}{36}$, $\mathbb{P}(X = 15) = \frac{2}{36}$, $\mathbb{P}(X = 16) = \frac{1}{36}$
- $\mathbb{P}(X = 18) = \frac{2}{36}$, $\mathbb{P}(X = 20) = \frac{2}{36}$, $\mathbb{P}(X = 24) = \frac{2}{36}$
- $\mathbb{P}(X = 15) = \frac{1}{36}$, $\mathbb{P}(X = 30) = \frac{2}{36}$, $\mathbb{P}(X = 36) = \frac{1}{36}$

And for any i not represented above has probability 0.

Question 3 (Section 4 Question 4)

5 men and 5 women are ranked according to their scores on an examination. Assume that no two scores are alike and all $10!$ possible rankings are equally likely. Let X denote the highest ranking achieved by a woman. (For instance, if $X = 1$ then the top-ranked person is female.) Find $\mathbb{P}(X = i), \quad i = 1, 2, 3, ..., 8, 9, 10.$

Solution

Consider the following cases:

1. If the top-ranked person is female, then we need to choose one of the women to be the number one $\binom{5}{1}$. Then, we rearrange the rest of the people, not caring about their scores $9!$. The probability is:

$$\mathbb{P}(X = 1) = \frac{\binom{5}{1}9!}{10!} = \frac{1}{2}$$

2. $X = 2$ represents the case that the highest ranking for a woman is second place (implying there's a man ranked first). First, we choose a man that did better $\binom{5}{1}$, then we choose the second best woman $\binom{5}{1}$. The placements of the rest of them don't matter. The probability is:

$$\mathbb{P}(X = 2) = \frac{\binom{5}{1}\binom{5}{1}8!}{10!} = \frac{5}{18}$$

3. Now two men dominate the top two spots $\binom{5}{2}$ however we don't know which man comes first before the other $2!$. Then we choose a woman to be 3rd place and the rest of them don't matter. The probability is:

$$\mathbb{P}(X = 3) = \frac{\binom{5}{2}2!\binom{5}{1}7!}{10!} = \frac{5}{36}$$

At this point in the course I expect students to understand (if you don't, then come to tutorials) so I will write the numeric solution for $X = 4, 5, ..., 10$. Note that:

$$\mathbb{P}(X = 4) = \frac{\binom{5}{3}3!\binom{5}{1}6!}{10!} = \frac{5}{84}$$

$$\mathbb{P}(X = 5) = \frac{\binom{5}{4}4!\binom{5}{1}5!}{10!} = \frac{5}{252}$$

$$\mathbb{P}(X = 6) = \frac{\binom{5}{5}5!\binom{5}{1}4!}{10!} = \frac{1}{252}$$

And we know that there's only 5 men and 5 women, so the rest of the probabilities should be 0. That is, $\mathbb{P}(X = 7) = \mathbb{P}(X = 8) = \mathbb{P}(X = 9) = \mathbb{P}(X = 10) = 0$.

Question 4 (Section 4 Questions 7-8)

Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:

(a) the maximum value to appear in the two rolls

(b) the minimum value to appear in the two rolls

(c) the sum of the two rolls

(d) the value of the first roll minus the value of the second roll

If the die are assumed as fair, calculate the probabilities associated with the random variables in parts (a) through (d).

Again, this question becomes simplier once you draw out the tables:

Greater Die Value		Outcome on Die 1					
		1	2	3	4	5	6
Outcome on Die 2	1	1	2	3	4	5	6
	2	2	2	3	4	5	6
	3	3	3	3	4	5	6
	4	4	4	4	4	5	6
	5	5	5	5	5	5	6
	6	6	6	6	6	6	6

Smaller Die Value		Outcome on Die 1					
		1	2	3	4	5	6
Outcome on Die 2	1	1	1	1	1	1	1
	2	1	2	2	2	2	2
	3	1	2	3	3	3	3
	4	1	2	3	4	4	4
	5	1	2	3	4	5	5
	6	1	2	3	4	5	6

Summation of the Die		Outcome on Die 1					
		1	2	3	4	5	6
Outcome on Die 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

Subtraction of the Die		Outcome on Die 1					
		1	2	3	4	5	6
Outcome on Die 2	1	0	1	2	3	4	5
	2	-1	0	1	2	3	4
	3	-2	-1	0	1	2	3
	4	-3	-2	-1	0	1	2
	5	-4	-3	-2	-1	0	1
	6	-5	-4	-3	-2	-1	0

Solution

Using the values of the table (and knowing the outcomes of the die), we can make the following observations:

(a) 1, 2, 3, ..., 6. Letting X_1 denote the maximum value to appear in the two rolls:

$$\mathbb{P}(X_1 = 1) = \frac{1}{36}, \ \mathbb{P}(X_1 = 2) = \frac{3}{36}, \ \mathbb{P}(X_1 = 3) = \frac{5}{36},$$

$$\mathbb{P}(X_1 = 4) = \frac{7}{36}, \ \mathbb{P}(X_1 = 5) = \frac{9}{36}, \ \mathbb{P}(X_1 = 6) = \frac{11}{36}$$

(b) 1, 2, ..., 5, 6. Letting X_2 denote the maximum value to appear in the two rolls:

$$\mathbb{P}(X_2 = 1) = \frac{11}{36}, \ \mathbb{P}(X_2 = 2) = \frac{9}{36}, \ \mathbb{P}(X_2 = 3) = \frac{7}{36},$$

$$\mathbb{P}(X_2 = 4) = \frac{5}{36}, \ \mathbb{P}(X_2 = 5) = \frac{3}{36}, \ \mathbb{P}(X_2 = 6) = \frac{1}{36}$$

(c) 2, 3, ..., 12. Letting X_3 denote the sum of the values appearing from the two rolls:

$$\mathbb{P}(X_3 = 2) = \frac{1}{36}, \ \mathbb{P}(X_3 = 3) = \frac{2}{36}, \ \mathbb{P}(X_3 = 4) = \frac{3}{36},$$

$$\mathbb{P}(X_3 = 5) = \frac{4}{36}, \ \mathbb{P}(X_3 = 6) = \frac{5}{36}, \ \mathbb{P}(X_3 = 7) = \frac{6}{36},$$

$$\mathbb{P}(X_3 = 8) = \frac{5}{36}, \ \mathbb{P}(X_3 = 9) = \frac{4}{36}, \ \mathbb{P}(X_3 = 10) = \frac{3}{36},$$

$$\mathbb{P}(X_3 = 11) = \frac{2}{36}, \ \mathbb{P}(X_3 = 12) = \frac{1}{36}$$

(d) -5, -4, ..., 5. Letting X_4 denote the value of the outcome on die 1 minus the outcome on die 2.

$$\mathbb{P}(X_4 = -5) = \frac{1}{36}, \ \mathbb{P}(X_4 = -4) = \frac{2}{36}, \ \mathbb{P}(X_4 = -3) = \frac{3}{36},$$

$$\mathbb{P}(X_4 = -2) = \frac{4}{36}, \ \mathbb{P}(X_4 = -1) = \frac{5}{36}, \ \mathbb{P}(X_4 = 0) = \frac{6}{36},$$

$$\mathbb{P}(X_4 = 1) = \frac{5}{36}, \ \mathbb{P}(X_4 = 2) = \frac{4}{36}, \ \mathbb{P}(X_4 = 3) = \frac{3}{36},$$

$$\mathbb{P}(X_4 = 4) = \frac{2}{36}, \ \mathbb{P}(X_4 = 5) = \frac{1}{36}$$

Unfortunately, I messed up and accidentally looked at the wrong section. I didn't want my time to go to waste, so I've just included it in this document.

Extra (Section 4 Self-Test Problems Question 1)

Suppose that the random variable X is equal to the number of hits obtained by a certain baseball player in his next 3 at bats. If:

- $\mathbb{P}(X = 1) = 0.3$
- $\mathbb{P}(X = 2) = 0.2$
- $\mathbb{P}(X = 0) = 3\mathbb{P}(X = 3)$

Find $\mathbb{E}[X]$.

Solution

From the question we have:

$$\sum_{i=0}^{3} \mathbb{P}(X = i) = 1 \quad \Rightarrow \quad \mathbb{P}(X = 0) + \mathbb{P}(X = 3) = \frac{1}{2}$$

Using the fact that $\mathbb{P}(X = 0) = 3\mathbb{P}(X = 3)$:

$$3\mathbb{P}(X = 3) + \mathbb{P}(X = 3) = \frac{1}{2} \quad \Rightarrow \quad \mathbb{P}(X = 3) = \frac{1}{8}, \;\; \mathbb{P}(X = 0) = \frac{3}{8}$$

Hence:

$$\mathbb{E}[X] = \sum_{i=0}^{3} i\times\mathbb{P}(X = i) = 0\times\mathbb{P}(X = 0)+1\times\mathbb{P}(X = 1)+2\times\mathbb{P}(X = 2)+3\times\mathbb{P}(X = 3)$$

$$= 0 + 0.3 + 0.4 + \frac{3}{8} = 1.075$$