

Question 1 (Section 4 Question 17)

Suppose that the distribution function of X is given by:

$$F_X(b) = \begin{cases} 0 & b < 0 \\ \frac{b}{4} & 0 \leq b < 1 \\ \frac{1}{2} + \frac{b-1}{4} & 1 \leq b < 2 \\ \frac{11}{12} & 2 \leq b < 3 \\ 1 & 3 \leq b \end{cases}$$

(a) Find $\mathbb{P}(X = i)$, $i = 1, 2, 3$.

(b) Determine $\mathbb{P}\left(\frac{1}{2} < X < \frac{3}{2}\right)$

Solution

(a) Allow us to compute using the definition...

$$\begin{aligned} \mathbb{P}(X = 1) &= \mathbb{P}(X \leq 1) - \mathbb{P}(X < 1) \\ &= \frac{1}{2} + \frac{(1) - 1}{4} - \frac{(1)}{4} = \frac{1}{4} \end{aligned}$$

$$\begin{aligned} \mathbb{P}(X = 2) &= \mathbb{P}(X \leq 2) - \mathbb{P}(X < 2) \\ &= \frac{11}{12} - \left(\frac{1}{2} + \frac{(2) - 1}{4}\right) = \frac{1}{6} \end{aligned}$$

$$\begin{aligned} \mathbb{P}(X = 3) &= \mathbb{P}(X \leq 3) - \mathbb{P}(X < 3) \\ &= 1 - \frac{11}{12} = \frac{1}{12} \end{aligned}$$

(b)

$$\begin{aligned} \mathbb{P}\left(\frac{1}{2} < X < \frac{3}{2}\right) &= \mathbb{P}\left(X < \frac{3}{2}\right) - \mathbb{P}\left(X < \frac{1}{2}\right) \\ &= \frac{1}{2} + \frac{(1.5) - 1}{4} - \frac{0.5}{4} = \frac{1}{2} \end{aligned}$$

Question 2 (Section 4 Question 18)

Four independent flips of a fair coin are made. Let X denote the number of heads obtained. Plot the probability mass function of the random variable $X - 2$.

Solution

Before diving into $X - 2$ let's find X . Since we're flipping four coins, we can either have 0, 1, 2, 3, or 4 heads appear. Now we need to find $\mathbb{P}(X = x)$.

1. To obtain 0 heads, we must obtain four tails in a row. Hence, $\mathbb{P}(X = 0) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.
2. To obtain 1 head, we first must choose which flip gives us a head. Then, we assign the the rest to be tails. This gives us $\mathbb{P}(X = 1) = \binom{4}{1} \left(\frac{1}{2}\right) \left(\frac{1}{2}\right)^3 = \frac{1}{4}$.
3. To obtain 2 heads, we choose two flips to provide us with heads. Then, we assign the the rest to be tails. This gives us $\mathbb{P}(X = 2) = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = \frac{3}{8}$.
4. The remaining cases follow a similar pattern so the explanation will be omitted. Hence:

$$\mathbb{P}(X = x) = \begin{cases} \frac{1}{16} & x = 0 \\ \frac{1}{4} & x = 1 \\ \frac{3}{8} & x = 2 \\ \frac{1}{4} & x = 3 \\ \frac{1}{16} & x = 4 \end{cases}$$

Let $Y = X - 2$. Then, note that:

$$\mathbb{P}(Y = y) = \mathbb{P}(X - 2 = y) = \mathbb{P}(X = y + 2)$$

So all we need to do is shift the values for y . This gives us:

$$\mathbb{P}(Y = y) = \begin{cases} \frac{1}{16} & y = -2 \\ \frac{1}{4} & y = -1 \\ \frac{3}{8} & y = 0 \\ \frac{1}{4} & y = 1 \\ \frac{1}{16} & y = 2 \end{cases}$$

Question 3 (Section 4 Question 25)

Two coins are to be flipped. The first coin will land on heads with probability .6, the second with probability .7. Assume that the results of the flips are independent, and let X equal the total number of heads that result.

- (a) Find $\mathbb{P}(X = 1)$
- (b) Determine $\mathbb{E}[X]$

Solution

- (a) For $X = 1$, either the first coin must be heads and the second must be tails 0.6×0.3 or the first coin must be tails and the second must be heads 0.4×0.7 . Hence, $\mathbb{P}(X = 1) = 0.46$
- (b) Now we need to find $\mathbb{P}(X = 0) = 0.4 \times 0.3 = 0.12$ (probability of obtaining no heads) and $\mathbb{P}(X = 2) = 0.6 \times 0.7 = 0.42$ (probability of both getting heads). Then,

$$\begin{aligned}\mathbb{E}[X] &= \sum_x xP(X = x) = 0 \times \mathbb{P}(X = 0) + 1 \times \mathbb{P}(X = 1) + 2 \times \mathbb{P}(X = 2) \\ &= 0 + 0.46 + 2 \times 0.42 = 1.3\end{aligned}$$

This means that in the long run, if one keeps on repeatedly flipping two unfair coins (the first having a 0.6 probability of heads, and the second with 0.7 probability of heads), on average you will see 1.3 heads every time you flip these two coins.

Question 4 (Section 4 Questions 28)

A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the expected number of defective items in the sample.

Solution

First, we need to compute the probability mass function. Let X denote the number of defective items that are picked.

1. If $X = 0$, then we need to pick 0 defective items. $\binom{16}{3} \div \binom{20}{3} = 0.4912281$
2. If $X = 1$, then we need to pick 1 defective items. $\binom{16}{2} \binom{4}{1} \div \binom{20}{3} = 0.4210526$
3. If $X = 2$, then we need to pick 2 defective items. $\binom{16}{1} \binom{4}{2} \div \binom{20}{3} = 0.08421053$
4. If $X = 3$, then we need to pick 3 defective items. $\binom{4}{3} \div \binom{20}{3} = 0.003508772$

Now we can happily compute the expected value.

$$\begin{aligned}\mathbb{E}[X] &= \sum_x xP(X = x) \\ &= 0 \times P(X = 0) + 1 \times P(X = 1) + 2 \times P(X = 2) + 3 \times P(X = 3) \\ &= 0 + [0.4210526] + 2 \times [0.08421053] + 3 \times [0.003508772] \\ &= 0 + 0.4210526 + 0.1684211 + 0.01052632 \\ &= 0.6\end{aligned}$$

Hence the expected value of defective items in the sample is 0.6. This means that in the long run, if one keeps on repeatedly choosing 3 items from a box of 20 containing 4 defective items, on average they will have 0.6 defective items each turn. Not bad!

Question 5 (Section 4 Questions 39)

If $\mathbb{E}[X] = 1, \mathbb{V}[X] = 5$ then find:

(a) $\mathbb{E}[(2 + X)^2]$

(b) $\mathbb{V}(4 + 3X)$

Relevant but Necessary Facts

1. $\mathbb{E}[X]$ represents a constant. Also, for any constant a , $\mathbb{E}[a] = a$.
2. For any probability mass function (or probability distribution) X, Y , we have that:

$$\begin{aligned}\mathbb{E}[X + Y] &= \sum_x \sum_y (x + y)P_{X,Y}(x + y) \\ &= \sum_x \sum_y (x)P_{X,Y}(x + y) + \sum_x \sum_y (y)P_{X,Y}(x + y) \\ &= \sum_x xP_X(x) + \sum_y (y)P_Y(y) = \mathbb{E}[X] + \mathbb{E}[Y]\end{aligned}$$

3. For any constant $a \in \mathbb{R}$:

$$\mathbb{E}[aX] = \sum_x (ax)P_X(x) = a \sum_x (x)P_X(x) = a\mathbb{E}[X]$$

4. The easily computable form of the variance is:

$$\begin{aligned}\mathbb{V}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + (\mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[2X\mathbb{E}[X]] + \mathbb{E}[(\mathbb{E}[X])^2] = \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2\end{aligned}$$

Additionally, by definition $\mathbb{V}(X) \geq 0$ so it should be clear that $\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$.

5. From above it's easy to show that $\mathbb{E}[X^2] = \mathbb{V}(X) + \mathbb{E}[X]^2$
6. For any constants a, b

$$\begin{aligned}\mathbb{V}(a + bX) &= \mathbb{E}[(a + bX - \mathbb{E}[a + bX])^2] = \mathbb{E}[(a + bX - (a + b\mathbb{E}[X]))^2] \\ &= \mathbb{E}[(a + bX - a - b\mathbb{E}[X])^2] = \mathbb{E}[(bX - b\mathbb{E}[X])^2] \\ &= \mathbb{E}[(b(X - \mathbb{E}[X]))^2] = \mathbb{E}[b^2(X - \mathbb{E}[X])^2] \\ &= b^2\mathbb{E}[(X - \mathbb{E}[X])^2] = b^2\mathbb{V}(X)\end{aligned}$$

Solution

Okay, now we're going to apply the previous facts. Teehee!

(a) Note that:

$$(2 + X)^2 = 4 + 4X + X^2$$

Furthermore, recall that $\mathbb{E}[X^2] = \mathbb{V}(X) + \mathbb{E}[X]^2 = 5 + 1^2 = 6$. Hence,

$$\begin{aligned}\mathbb{E}[(2 + X)^2] &= \mathbb{E}[4 + 4X + X^2] \\ &= \mathbb{E}[4] + \mathbb{E}[4X] + \mathbb{E}[X^2] \\ &= 4 + \mathbb{E}[2X] + \mathbb{E}[X^2] \\ &= 4 + 4\mathbb{E}[X] + 6 = 10 + 4 = 14\end{aligned}$$

(b) Behold:

$$\begin{aligned}\mathbb{V}(4 + 3X) &= \mathbb{V}(3X) \\ &= 3^2\mathbb{V}(X) \\ &= 9(5) = 45\end{aligned}$$