Question 1 (Section 4 Question 40)

A ball is drawn from an urn containing 3 white and 3 black balls. After the ball is drawn, it is replaced and another ball is drawn. This process goes on indefinitely. What is the probability that of the first 4 balls drawn, exactly 2 are white?

Solution

This should be treated as a warm-up from the midterm recess (or what most Canadian universities call reading week.) Since we have replacement, we know that there's equal $\frac{3}{6} = \frac{1}{2}$ chance of getting either white or black, and we just need to pick the 2 placements out of the 4 for a white ball to occur (and then we know the rest must be black.) Hence, the probability is:

$$\left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 \binom{4}{2} = \frac{3}{8}$$

Question 2 (Section 4 Question 50)

When coin 1 is flipped, it lands on heads with probability 0.4; when coin 2 is flipped, it lands on heads with probability 0.7. One of these coins is randomly chosen and flipped 10 times.

(a) What is the probability that the coin lands on heads on 7 exactly of the 10 flips?

(b) Given that the first of these 10 flips lands heads, what is the conditional probability that exactly 7 of the 10 flips land on heads?

Solution

For convenience, let A denote the event of obtaining heads 7 out of 10 times. Let C_1 denote the event of flipping coin 1, and C_2 denote the event of flipping coin 2.

(a) Assuming there's an equal chance of picking any of the coins, first we compute the probability of getting heads 7 times given we flip coin 1. Then, using the total law of probability:

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}(A \cap C_1) + \mathbb{P}(A \cap C_2)\\ &= \mathbb{P}(C_1)\mathbb{P}(A|C_1) + \mathbb{P}(C_2)\mathbb{P}(A|C_2)\\ &= (0.5) \times \left[(0.4)^7(0.6)^3\binom{10}{7}\right] + (0.5) \times \left[(0.7)^7(0.3)^3\binom{10}{7}\right]\\ &= 0.1546476\end{aligned}$$

(b) The main difference is that if we already know 1 of them lands heads (being the first), then we need to ensure 6 of the remaining 9 flips shall be heads. Then, now let B denote the event of obtaining heads 6 out of 9 times. Thus:

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B \cap C_1) + \mathbb{P}(B \cap C_2)\\ &= \mathbb{P}(C_1)\mathbb{P}(B|C_1) + \mathbb{P}(C_2)\mathbb{P}(B|C_2)\\ &= (0.5) \times \left[(0.4)^6(0.6)^3\binom{9}{6}\right] + (0.5) \times \left[(0.7)^6(0.3)^3\binom{9}{6}\right]\\ &= 0.1705729\end{aligned}$$

As expected, the probability increases.

Question 3 (Section 4 Question 60)

Suppose that the number of accidents occurring on a highway each day is a Poisson random variable with parameter $\lambda = 3$.

(a) Find the probability that 3 or more accidents occur today.

(b) Repeat part (a) under the assumption that at least 1 accident occurs today.

Solution

Again, for ease let A represent the event where 3 or more accidents occur.

(a) As a result, A^c represents the event that less than 3 accidents occur (either none, 1, or 2).

$$\begin{aligned}
\mathbb{P}(A) &= 1 - \mathbb{P}(A^c) \\
&= 1 - (\mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2)) \\
&= 1 - (\frac{e^{-3}(3)^0}{0!} + \frac{e^{-3}(3)^1}{1!} + \frac{e^{-3}(\lambda)^2}{2!}) \\
&= 1 - (0.4231901) \\
&= 0.5768099
\end{aligned}$$

(b) Let B represent the event where at least 1 accident occurs today. Then:

$$\begin{aligned}
\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\
&= \frac{\mathbb{P}(A)}{1 - \mathbb{P}(B^c)} \\
&= \frac{0.5768099}{1 - (\mathbb{P}(X = 0))} \\
&= \frac{0.5768099}{1 - 0.04978707} \\
&= \frac{0.5768099}{0.9502129} \\
&= 0.6070323
\end{aligned}$$

Question 4 (Section 4 Questions 68)

Each of 500 soldiers in an army company independently has a certain disease with probability $(\frac{1}{10^3})$. This disease will show up in a blood test, and to facilitate matters, blood samples from all 500 soldiers are pooled and tested.

(a) What is the (approximate) probability that the blood test will be positive (that is, at least one person has the disease)?

(b) Suppose now that the blood test yields a positive result. What is the probability, under this circumstance, that more than one person has the disease?

(c) Now, suppose one of the 500 people is Jones, who knows that he has the disease. What does Jones think is the probability that more than one person has the disease?

(d) Because the pooled test was positive, the authorities have decided to test each individual separately. The first $(i-1)$ of these tests were negative, and the ith one–which was on Jones – was positive. Given the preceding scenario, what is the probability, as a function of i, that any of the remaining people have the disease?

Solution: a, b

(a) Here it asks for the approximate probability. Since n is large and the probability of getting infected is small, we use the Poisson approximation. Let D denote the number of people who are diseased. Using the Poisson approximation, $np = 500(0.001) = 0.5 \quad \Rightarrow \quad D \sim Poisson(\lambda = 0.5)$. Then,

$$\mathbb{P}(D \geq 1) = 1 - \mathbb{P}(D = 0) = 1 - \frac{e^{-1/2}(1/2)^0}{0!} = 0.3934693$$

(b) For the blood test to be positive, at least one person must be diseased; now we're checking if there are at least two.

$$\mathbb{P}(D \geq 2 | D \geq 1) = \frac{\mathbb{P}(D \geq 2 \cap D \geq 1)}{\mathbb{P}(D \geq 1)} = \frac{\mathbb{P}(D \geq 2)}{\mathbb{P}(D \geq 1)}$$

$$\mathbb{P}(D \geq 2) = 1 - \mathbb{P}(D = 0) - \mathbb{P}(D = 1) = 1 - e^{-1/2} - \frac{e^{-1/2}(1/2)^1}{1!} = 0.09020401$$

$$\frac{\mathbb{P}(D \geq 2)}{\mathbb{P}(D \geq 1)} = \frac{0.09020401}{0.3934693} = 0.229253$$

Solution: c, d

(c) If we can identify who exactly is diseased, then we must check for the remainder 499 soldiers. Here, let D_0 denote the number of people who are diseased, but with a maximum of 499 soldiers. Using the Poisson approximation, $np = 499(0.001) = 0.499 \quad \Rightarrow \quad D \sim Poisson(\lambda = 0.499)$. Then,

$$\mathbb{P}(D_0 \geq 1) = 1 - \mathbb{P}(D_0 = 0) = 1 - \frac{e^{-0.499}(0.499)^0}{0!} = 0.3928625$$

(d) Since Jones is the ith soldier, which is the first to be identified as positive for the disease. This is analogous to a "successful" event. We realise that there are $(i-1)th$ soldiers that are tested as negative: this is analogous to a "failure". This resembles a geometric distribution. Let J denote the event of which soldier out of the 500 it takes to finally find someone that is diseased. Here, we have that $J \sim Geometric(p = 0.001)$. However, we also want to see if more than just Jones is diseased. Hence,

$$\mathbb{P}(D \geq 2 | J = i) = \frac{\mathbb{P}(D \geq 2, J = i)}{\mathbb{P}(J = i)}$$

Note that $D \geq 2, J = i$ represents looking for the probability that someone else except John is infected, and we know that the first $i - 1$ participants tested as negative. Hence, we have $500 - i$ soldiers remaining to test and we just need at least one of them. For short, let R represent the event where after John, the number of remaining soldiers that are tested as positive. Here, $R \sim Binom(n = 500 - i, p = 0.001)$ and therefore:

$$\frac{\mathbb{P}(R \geq 1)}{\mathbb{P}(J = i)} = \frac{1 - \mathbb{P}(R = 0)}{(0.001)(0.999)^{i-1}}$$

$$= \frac{1 - \binom{500-i}{0}(0.001)^0(0.999)^{500-i}}{(0.001)(0.999)^{i-1}} = \frac{1 - (0.999)^{500-i}}{(0.001)(0.999)^{i-1}}$$

Side note: technically, if you want to be consistent with the Poisson approximation (like parts a, b) one could technically use $1 - e^{-(500-i)(0.001)}$ instead for the numerator.

Question 5 (Section 4 Questions 61 a,b)

Compare the Poisson approximation with the correct binomial probability for the following cases:

(a) $\mathbb{P}(X = 2)$ when $n = 8, p = 0.1$

(b) $\mathbb{P}(X = 9)$ when $n = 10, p = 0.95$

Solution

Typically you want the use the Poisson approximation $\lambda = np$ when n is large and p is small. Hence, this exercise wants us to compare the results...

(a) First we will compute the Poisson approximation:

$$\lambda = np = (8)(0.1) = 0.8, \quad \mathbb{P}(X = 2) = \frac{e^{-0.8}(0.8)^2}{2!} = 0.1437853$$

Now here is the correct binomial probability:

$$\mathbb{P}(X = 2) = \binom{8}{2}(0.1)^2(0.9)^6 = 0.1488035$$

Not that bad of an approximation.

(b) We expect something else since p is large now. Witness this approximation:

$$\lambda = np = (10)(0.95) = 9.5, \quad \mathbb{P}(X = 9) = \frac{e^{-9.5}(9.5)^9}{9!} = 0.1300025$$

Now here is the correct binomial probability:

$$\mathbb{P}(X = 9) = \binom{10}{9}(0.95)^9(0.05)^1 = 0.3151247$$

As we expected, we get a pretty bad approximation.