Tutorial 8

Question 1 (Section 5 Question 4)

The probability density function of X, the lifetime of a certain type of electronic device (measured in hours), is given by:

$$f_X(x) = \begin{cases} \frac{10}{x^2} & x > 10 \\ 0 & x \le 10 \end{cases}$$

(a) Find $\mathbb{P}(X > 20)$.

(b) What is the cumulative distribution function of X?

(c) What is the probability that of 6 such types of devices, at least 3 will function for at least 15 hours? What assumptions are you making?

a) $\mathbb{P}(X>20) = \int_{20}^{\infty} 10x^{-2}\,dx = \frac{-10}{x} + c \Big|_{20}^{\infty} = \lim_{x\to\infty} \frac{-10}{x} + c - \left(\frac{-10}{20} + c\right) = \frac{1}{2}$

b) $F_X(x) = \int_{10}^{x} 10t^{-2}\,dt = \frac{-10}{t} + c \Big|_{10}^{x} = \frac{-10}{x} + c - (-1 + c) = 1 - 10/x$

So, $F_X(x) = \begin{cases} 0 & x \le 10 \\ 1 - 10/x & x > 10 \end{cases}$ ✱ NEVER forget the support!

c) Let Y denote the # of devices that last at least 15 hours. So, $Y \sim \text{Binom}(n=6,\ p=\mathbb{P}(X\ge 15))$

(Making the assumption that the lifetime of the devices are independent of each other.)

Then, $\mathbb{P}(X\ge15) = 1-\mathbb{P}(X<15) = 1 - F_X(15) = 1 - \left(1 - \frac{10}{15}\right) = 2/3$

and $\mathbb{P}(Y\ge3) = 1 - \mathbb{P}(Y\le2) = 1 - [\mathbb{P}(Y=0) + \mathbb{P}(Y=1) + \mathbb{P}(Y=2)]$

$= 1 - \left[\binom{6}{0}(2/3)^0(1/3)^6 + \binom{6}{1}(2/3)^1(1/3)^5 + \binom{6}{2}(2/3)^2(1/3)^4\right]$

$= 1 - (0.1001372) = 0.8998628$

Question 2 (Section 5 Question 5)

A filling station is supplied with gasoline once a week. If its weekly volume of sales in thousands of gallons is a random variable with probability density function:

$$f_X(x) = \begin{cases} 5(1-x)^4 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

what must the capacity of the tank be so that the probability of the supply being exhausted in a given week is .01?

Want to solve c for: $\mathbb{P}(X>c) = 0.01$

$\mathbb{P}(X>c) = \int_c^1 5(1-x)^4\,dx$ do: u-sub. Let $u = 1-x$, $du = dx$ (do bounds at the end)

$= \int 5u^4\,du = \frac{5u^5}{5} + \text{const.}$

$= (1-x)^5 + \text{const} \Big|_c^1 = (1-c)^5 \overset{\text{set}}{=} 0.01$

$\Rightarrow c = 1 - (0.01)^{1/5} = 0.6018928$

Hence the capacity should be 0.6018928 thousands of gallons.

(or: 601.8928 gallons.)

Question 3 (Section 5 Question 6 a, b)

Compute $\mathbb{E}[X]$ if X has a density function given by:

(a) $$f_X(x) = \begin{cases} \frac{1}{4}xe^{-x/2} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) $$f_X(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $E(X) = \int_0^\infty x\left(\frac{1}{4}xe^{-x/2}\right)dx = \boxed{\int_0^\infty \frac{1}{4}x^2e^{-x/2}\,dx}$

2 methods: 1. use IBP
2. use known distributions ↑ better method
} both are long.

Compare w/ the Gamma(α, λ) pdf:

$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{(\alpha-1)} e^{-\lambda x}$ try: $\alpha - 1 = 2 \Rightarrow \alpha = 3$, $\lambda = 1/2$

Recall: $\Gamma(n) = (n-1)!$ $\Gamma(3) = 2! = 2$

$\lambda^\alpha = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$ Hence $\frac{\lambda^\alpha}{\Gamma(\alpha)} = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$

So, $\int_0^\infty \frac{1}{16}x^2e^{-\frac{1}{2}x}\,dx = 1$ multiply both sides by 4:

$\boxed{\int_0^\infty \frac{1}{4}x^2e^{-x/2}\,dx} = 4 \Rightarrow E(X) = 4.$

b) $E(X) = \int_{-1}^1 x\,c(1-x^2)\,dx = c\int_{-1}^1 x - x^3\,dx = c\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_{x=-1}^{x=1}$

$= c\left(\frac{1}{2} - \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{4}\right)\right] = c\left(\frac{1}{2} - \frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right) = 0$

Question 4 (Section 5 Question 13)

You arrive at a bus stop at 10 a.m., knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

$X \sim \text{Unif}(a,b)$: $f_X(x) = \frac{1}{b-a}$ $a<x<b$

a) $\mathbb{P}(X>10) = \int_{10}^{30} \frac{1}{30-0}\,dt = \frac{1}{30}t + c \Big|_{10}^{30} = 3 - 1/3 = 2/3$

b) $\mathbb{P}(X>25 \mid X>15) = \frac{\mathbb{P}(X>25)}{\mathbb{P}(X>15)} = \frac{1/6}{1/2} = 1/3$

$\mathbb{P}(X>15) = t/30 + c \Big|_{15}^{30} = 1 - 1/2 = 1/2$

$\mathbb{P}(X>25) = t/30 + c \Big|_{25}^{30} = 1 - 5/6 = 1/6$

Previously Question 1 (Section 5 Question 1)

This question was removed due to time constraints. However, the solution was already written so we provided this. Let X be a random variable with probability density function:

$$f_X(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of c?

(b) What is the cumulative distribution function of X?

a) Note that for a valid pdf, $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Hence, $1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} c(1-x^2)\,dx + \int_1^\infty 0\,dx$

$\Rightarrow 1 = c\left(x - \frac{x^3}{3}\right) + \text{constant}\Big|_{-1}^{1} \Rightarrow 1 = c\left[1 - \frac{1}{3} - \left(-1 - \frac{(-1)}{3}\right)\right] \Rightarrow 1 = 4/3\,c$

$\Rightarrow c = 3/4$

b) $F_X(x) = \mathbb{P}(X \le x) = \int_{-1}^{x} 1/2(1-t^2)\,dt = \frac{1}{2}(t - t^3/3) + c\Big|_{-1}^{x}$

$= 3/4\left[x - \frac{x^3}{3} - \left(-1 - \frac{(-1)}{3}\right)\right] = 3/4\left[x - \frac{x^3}{3} + \frac{2}{3}\right]$

Hence, $F_X(x) = \begin{cases} 0 & x \le -1 \\ 3/4\left(x - \frac{x^3}{3} + \frac{2}{3}\right) & -1 < x < 1 \\ 1 & x \ge 1 \end{cases}$

NEVER forget the support! ^_^